EXTERNALLY CURVED EUCLIDEAN SPACES

ARMIN HAKEN

1. Introduction

Euclidean spaces are "flat" without Gaussean curvature but their extrinsic curvature can be mathematically quite interesting. This paper is intended to draw attention to open questions about the curvature of n-dimensional Euclidean space embedded in m-dimensional space. The cases $\{n = 2, m = 3\}$ and $\{n = 1\}$ are familiar. However, for $\{n \geq 2, m \geq 4\}$ it seems that the common characterization of curvature as a scalar has led everyone to assume that the geometry is trivial. Once curvatures are characterized as vectors, it leads to field lines whose geometry looks to be quite interesting yet largely unexplored. This paper just gives motivation and states some conjectures and questions but gives no proofs.

2. One-Dimensional Space

The case of $\mathbb{E}^1$ embedded in $\mathbb{E}^m$ is just a curve parameterized by length running through m-dimensional Euclidean space. As long as the curve is never straight, its shape is completely determined by its curvature and the higher derivatives of the trajectory of a point traversing the curve [1]. Note that the direction of curvature and torsion is always relative to the curve and not specified using the coordinate system of the embedding space. Only the magnitude of the curvature is of interest so there is no reason to treat it as a vector.

It is also worth mentioning that embedding $\mathbb{E}^1$ in $\mathbb{E}^3$ compared to $\mathbb{E}^2$ gives more interesting shapes (spirals and knots). Once we go to $\mathbb{E}^4$ and higher there no longer are knots.

3. Two-Dimensional Space in $\mathbb{E}^3$

Two-dimensional surfaces in three-space are extensively studied [2]. At each point, extrinsic curvature can only point in one direction so it is a scalar quantity. At each point there are directions of *principal curvature* in which the curvature is maximized/minimized. When the surface is Euclidean, it can curve only in one principal direction and has to be straight in the orthogonal direction. If the surface is infinite its principal directions form an orthogonal grid and globally there always a direction in which the surface has to be straight. Famously, if we hold a slice of pizza in a curved shape, it will not droop down along the spine of the curve. We could curve our entire 3D space downward into a fourth dimension and then the pizza would droop along with the entire room.

Date: April 10, 2025.
The latest update of this document can be found at http://www.asi9.com.
Email: arminhaken@asi9.com

4. Two-Dimensional Space in Higher Dimensions

It seems like surfaces embedded in higher dimensions have been neglected [3]. Once a two-dimensional surface is embedded in $\mathbb{E}^4$ or higher, the space normal to the surface is multi-dimensional. We pay attention to the *direction* as well as the magnitude of the curvature by treating it as a *vector*. In the case of $\mathbb{E}^1$ embedded in higher-dimensional spaces there is only one curvature associated with each point so it is traditional to focus just on the magnitude of that curvature (up to a change of sign). For the curve there can be no confusion about different directions for different curvatures at a point. On the two-dimensional surface there are multiple curvatures associated with every point depending on which direction we face on the surface. The directions of these curvatures within the orthogonal space become relevant.

5. Theorema Egregium

For the case of $\mathbb{E}^2$ in $\mathbb{E}^3$, Gauss' *Theorema Egregium* specifies that at each point the curvature in one principal direction times the curvature in the other (orthogonal) principal direction is 0. Once the surface is embedded in higher dimensions and curvatures are characterized as vectors, the theorema can be extended to say that for a (flat) Euclidean space if the principal curvature directions are orthogonal to each other their associated external curvatures are also orthogonal. The directions of principal curvatures are not always obvious so we also consider the *torsion* that accompanies the curvature in non-principal directions to get a statement that does not rely on principal curvature directions.

6. Torsion and Warp

Assume a coordinate system on $\mathbb{E}^2$ so that at the origin there is maximum positive curvature in the x direction and 0 curvature in the y direction. Locally at the origin the space is shaped like a cylinder curving around the y axis. Now consider a line through the origin that is not horizontal or vertical. That line has some positive curvature but also torsion. Locally it has the shape of a helix inscribed in the curved space. Depending on the angle of the line it has more or less torsion. The line $y = x$ has the most torsion and the line $y = -x$ has the same magnitude of torsion but in the opposite direction. It is tempting to think of the torsion as clockwise or counterclockwise but in higher dimensions it may be more complicated than that.

Quantitatively, assume the line makes angle θ with the x axis and κ is the curvature along the x axis. The projected curvature (at the origin) along our line is a vector perpendicular to the space with length $\kappa \cos^2(\theta)$. The torsion on the line (at the origin) is a vector of length $\kappa \sin(\theta) \cos(\theta)$. We assume the direction of the torsion vector is outside of the Euclidean embedded space. The *warp vector* associated with the line (at the origin) is the sum of the curvature and torsion vectors. If we calculate the warp along the perpendicular line through the origin (at an angle of $\theta + \pi/2$) we find that the second warp vector is perpendicular to the first one.

We can say that in $\mathbb{E}^2$ if we take any two perpendicular lines, the two warp vectors associated with the intersection point have dot product 0.

7. Orthogonal Warp Principle

The higher-dimensional analog to the zero Gaussian curvature case of Theorema Egregium is the Orthogonal Warp Principle.

Let P be a point in the space $\mathbb{E}^n$ $(n > 1)$ embedded in $\mathbb{E}^m$. Let $\vec{x_1}, \vec{x_2}, \ldots\ \vec{x_n}$ be mutually orthogonal vectors starting from P (in the n-dimensional tangent space at P). The $\vec{x_i}$ vectors represent a Cartesian coordinate system in $\mathbb{E}^n$ with origin at P. For some $q \leq m-n$ let $\vec{c_1}, \vec{c_2}, \ldots\ \vec{c_q}$ be mutually orthogonal external curvature vectors at P. These $\vec{c_i}$ vectors are all normal to $\mathbb{E}^n$ at P. The principal curvature directions at P corresponding to the $\vec{c_i}$ are not necessarily orthogonal to each other. For each pair $\vec{x_i}, \vec{c_j}$ let $\vec{k_{i,j}}$ be the projected curvature of $\vec{c_j}$ onto $\vec{x_i}$. Let $\vec{t_{i,j}}$ be the torsion induced on $\vec{x_i}$ by $\vec{c_j}$. The torsion vectors live in some separate q dimensional space so they are orthogonal to the curvature vectors. The torsion vectors $\vec{t_{i,j}}$ and $\vec{t_{i',j'}}$ are orthogonal for $j \neq j'$ and parallel for $j = j'$. For each $i \in (1\ldots\ n)$ the warp vector $\vec{w_i}$ is defined to be $\sum_{j=1}^{q} (\vec{c_{i,j}} + \vec{t_{i,j}})$.

Because $\mathbb{E}^n$ is intrinsically flat, $\vec{w_i} \cdot \vec{w_{i'}} = 0 \mid i \neq i'$. The warp vectors are pairwise orthogonal or $\vec{0}$. (Certainly if $m < 2n$ and we happen to have picked principal curvature directions without torsion, then some of the warps must be $\vec{0}$.)

8. Field Lines

When $\mathbb{E}^n$ is embedded in $\mathbb{E}^m$ there can be several mutually orthogonal external curvatures at each point in $\mathbb{E}^n$. Each of these curvatures has a principal direction but these principal directions need not be perpendicular to each other. Following such a principal direction will generally trace out a curve within $\mathbb{E}^n$ which we call a *field line*. In the case of an infinite two dimensional sheet embedded in three dimensions the field lines need to be straight. If the sheet is allowed to curve into the shape of a cone the field lines curve. For a cone we can calculate the curvature of the field lines from the external curvature on the sheet and its directional derivatives.

Another example of curved field lines comes from taking a two dimensional sheet and curving it into a shape that is embedded in $\mathbb{E}^3$. In the 3-dimensional space this perhaps looks like a cylinder or a corrugated sheet of steel. Now imagine that the entire enclosing $\mathbb{E}^3$ is curved into a fourth dimension while remaining “flat”. Within $\mathbb{E}^3$ all the curvature field lines point in the same direction. From the point of view of the embedded $\mathbb{E}^2$ cylinder, the projections of these field lines typically no longer all point in the same direction. This gives an example of $\mathbb{E}^2$ embedded in $\mathbb{E}^4$. The field lines from the original curvature into the third dimension are straight and the the field lines from the additional curvature into the fourth dimension are curved. In general, what are the constraints on the curvatures of these field lines? Can we calculate the curvature of the field lines based on the external curvatures and their derivatives?

9. Slices

When $\mathbb{E}^n$ is embedded in $\mathbb{E}^m$ we can also investigate Euclidean "flat" subspaces of $\mathbb{E}^n$ and ask about the curvature of such *slices*. We have $\mathbb{E}^q$ embedded in $\mathbb{E}^n$ without any curvature but $\mathbb{E}^q$ is curved within $\mathbb{E}^m$. We see that the directions of principal curvature from $\mathbb{E}^n$ need not lie within $\mathbb{E}^q$ even though in some sense all the curvature of $\mathbb{E}^q$ is inherited from $\mathbb{E}^n$. Studying two-dimensional slices of a curved higher-dimensional Euclidean space seems like a slightly easier way to "visualize" the curvature of the space, provided we have a characterization of what can happen to a "flat" surface under multi-dimensional external curvature.

10. Euclidean Topology

Is there anything potentially interesting about a variant of topology that does not permit stretching? A topological question that comes to mind is the status of higher-dimensional knots; embedded "flat" surfaces that cannot be transformed into each other without self-intersections. The transformation in this case would consist entirely of changes in extrinsic curvature.

References

[1] R. Sulanke, "The Fundamental Theorem for Curves in the n-Dimensional Euclidean Space.." `https://www2.mathematik.hu-berlin.de/~sulanke/diffgeo/euklid/ECTh.pdf`, 2020.

[2] M. Berger, *A Panoramic View of Riemannian Geometry.* Springer, 2003.

[3] M. Berger, *A Panoramic View of Riemannian Geometry*, ch. 1.6, p. 29. Springer, 2003. "see why for yourself".