# EXTERNALLY CURVED EUCLIDEAN SPACES

ARMIN HAKEN

## 1. Introduction

This paper is intended to draw attention to some questions about the curvature of $n$-dimensional Euclidean space embedded in $m$-dimensional space. The cases $\{n = 2, m = 3\}$ and $\{n = 1\}$ have been extensively studied. However, for $\{n \geq 2, m \geq 4\}$ the geometry seems to be interesting yet largely unexplored. The paper states some conjectures and questions but gives no proofs.

## 2. One-Dimensional Space

The case of $\mathbb{E}^1$ embedded in $\mathbb{E}^m$ is just a curve parameterized by length running through $m$-dimensional Euclidean space. As long as the curve is never straight, its shape is completely determined by its curvature and the higher derivatives of the trajectory of a point traversing the curve [2]. Note that the direction of curvature and torsion is always relative to the curve and not specified using the coordinate system of the embedding space.

It is also worth mentioning that embedding $\mathbb{E}^1$ in $\mathbb{E}^3$ compared to $\mathbb{E}^2$ gives more interesting shapes (spirals and knots). Once we go to $\mathbb{E}^4$ and higher there no longer are knots.

## 3. Two-Dimensional Space in $\mathbb{E}^3$

Two-dimensional surfaces in three-space are extensively studied [1]. At each point there are directions of *principal curvature* in which the curvature is maximized/minimized. When the surface is Euclidean, it can curve only in one principal direction and has to be straight in the orthogonal direction. If the surface is infinite its principal directions form an orthogonal grid and globally there is one direction in which the shape of the surface has to be straight. Famously, if we hold a slice of pizza in a curved shape, it will not droop down along the spine of the curve. If a fourth dimension were available, the pizza would droop into that dimension but in 3D there is a nice non-local property of rigidity. We can focus on the magnitude of the curvature and avoid mentioning its direction since at every point there is only one direction that is orthogonal to the surface.

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Email: arminhaken@asi9.com

## 4. Two-Dimensional Space in Higher Dimensions

Once a two-dimensional surface is embedded in $\mathbb{E}^4$ or higher we have to pay attention to the direction as well as the magnitude of the curvature. At each point on the surface the orthogonal space is multi-dimensional so the curvature should be considered a vector. For the case of $\mathbb{E}^1$ embedded in higher dimensional spaces there is only one curvature associated with each point so it is ok to focus just on the magnitude of that curvature (up to a change of sign). For the curve there can be no confusion about different directions for different curvatures at a point. On the two-dimensional surface there are multiple curvatures associated with every point depending on which direction we face on the surface. The external directions of these curvatures become relevant.

## 5. Theorema Egregium

For the case of $\mathbb{E}^2$ in $\mathbb{E}^3$, Gauss' *Theorema Egregium* specifies that at each point the curvature in one principal direction times the curvature in the other (orthogonal) principal direction is 0. In higher dimensions the directions of principal curvature are not obvious so we consider *torsion* as well as curvature.

## 6. Torsion and Warp

Assume a coordinate system on $\mathbb{E}^2$ so that at the origin there is maximum positive curvature in the $x$ direction and 0 curvature in the $y$ direction. Locally at the origin the space is shaped like a cylinder curving around the $y$ axis. Now consider a line through the origin that is not horizontal or vertical. That line has some positive curvature but also torsion. Locally it has the shape of a helix inscribed in the curved space. Depending on the angle of the line it has more or less torsion. The line $y = x$ has the most torsion and the line $y = -x$ has the same magnitude of torsion but in the opposite direction. It is tempting to think of the torsion as clockwise or counterclockwise but more accurate to think of it pointing in the $x$ or $-x$ direction. If the curvature were along the $y$ direction instead of the $x$ direction the torsion would then be in the $y$ direction.

Quantitatively, assume the line makes angle $\theta$ with the $x$ axis and $\kappa$ is the curvature along the $x$ axis. The curvature (at the origin) along our line is a vector perpendicular to the space with length $\kappa \cos^2(\theta)$. The torsion on the line (at the origin) is a vector in the $x$ direction with length $\kappa \sin(\theta)\cos(\theta)$. The *warp vector* along the line (at the origin) is the sum of the curvature and torsion vectors. If we calculate the warp along the perpendicular line through the origin (at an angle of $\theta + \pi/2$) we find that the second warp vector is perpendicular to the first one.

We can say that in $\mathbb{E}^2$ if we take any two perpendicular lines, the two warp vectors associated with the intersection point have dot product 0.

## 7. Orthogonal Warp Principle

The higher dimensional analog to the Theorema Egregium is the Orthogonal Warp Principle.

Let $P$ be a point in the space $\mathbb{E}^n$ embedded in $\mathbb{E}^m$. Let $\vec{x_1}, \vec{x_2}, ...\vec{x_n}$ be mutually orthogonal vectors starting from $P$ (in the $n$-dimensional tangent space at $P$). Let $\vec{c_1}, \vec{c_2}, ...\vec{c_n}$ be the curvatures at $P$ for the directions $\vec{x_1}, \vec{x_2}, ...\vec{x_n}$ respectively. These vectors are all perpendicular to $\mathbb{E}^n$ at $P$. Let $\vec{t_1}, \vec{t_2}, ...\vec{t_n}$ be the associated torsion vectors. The torsion vectors are each perpendicular to the curvature vectors. The warp vectors $\vec{w_1}, \vec{w_2}, ...\vec{w_n}$ are $\vec{c_1} + \vec{t_1}, \vec{c_2} + \vec{t_2}, ...\vec{c_n} + \vec{t_n}$ respectively.

Because $\mathbb{E}^n$ is intrinsically flat, $\vec{w_i} \cdot \vec{w_j} = 0 \mid i \neq j$. The warp vectors are pairwise orthogonal or $\vec{0}$. (Certainly if $m < 2n$ and we happen to have picked principal curvature directions without torsion, then some of the warps must be $\vec{0}$.)

## 8. Frames and Field Lines

When $\mathbb{E}^n$ is embedded in $\mathbb{E}^m$ and $m \leq 2n$ there will generally be principal curvature directions within $\mathbb{E}^n$, directions in which the warp is purely curvature without torsion. At every point the mutually orthogonal directions of principal curvature form a *frame*. As the point moves, the frame might rotate and twist. If we move in one of the principal directions, the frame might rotate a bit so that principal direction traces out an arc in $\mathbb{E}^n$. We call such an arc a *field line*. For example, on the surface of a cone, the field lines are straight lines toward the apex of the cone and circular arcs perpendicular to the straight lines. The field lines curve when there is a gradient in the extrinsic curvature in the direction of the field line. If the extrinsic curvature of the space along field line $F$ is $\vec{c}$ and the gradient of curvatures along lines parallel to $F$ is $F^\perp$, the curvature of $F$ itself within the space is $\|\vec{c}\| F^\perp$. In general these field lines can weave through the space while intersecting at right angles. It would be nice to characterize more precisely what these field lines can do.

## 9. Slices

When $\mathbb{E}^n$ is embedded in $\mathbb{E}^m$ we can also investigate Euclidean "flat" subspaces of $\mathbb{E}^n$ and ask about the curvature of such *slices*. We have $\mathbb{E}^q$ embedded in $\mathbb{E}^n$ without any curvature but $\mathbb{E}^q$ is curved within $\mathbb{E}^m$. We see that the directions of principal curvature from $\mathbb{E}^n$ need not lie within $\mathbb{E}^q$ even though in some sense all the curvature of $\mathbb{E}^q$ is inherited from $\mathbb{E}^n$. Studying two dimensional slices of a curved higher dimensional Euclidean space seems like a slightly easier way to "visualize" the curvature of the space.

## 10. Topology

A topological question that comes to mind is the status of higher-dimensional knots; embedded Euclidean surfaces that cannot be transformed into each other without self-intersections. The transformation in this case would consist of changes in curvature.

## References

[1] Berger, M. *A Panoramic View of Riemannian Geometry.* Springer, 2003.

[2] Sulanke, R. The Fundamental Theorem for Curves in the n-Dimensional Euclidean Space. http://www-irm.mathematik.hu-berlin.de/~sulanke/diffgeo/euklid/ECTh.pdf, 2020.