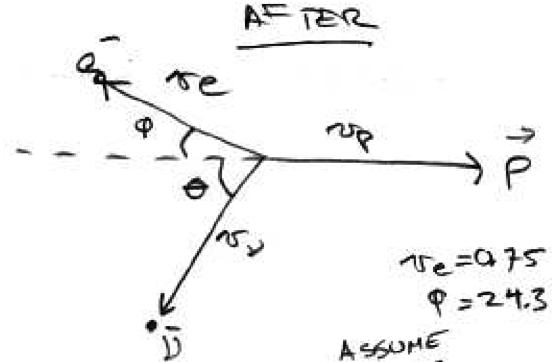
A260 HW #2

1

BEFORE

$$\underset{\sim}{P} = \begin{bmatrix} m_n \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

AFTER



$v_e = 0.75$
$\varphi = 24.3°$

ASSUME MASSLESS

$$\underset{\sim}{P} = \underset{\sim}{P_e} + \underset{\sim}{P_\nu} + \underset{\sim}{P_p}$$

$$= \begin{bmatrix} \frac{m_e}{\sqrt{1-v_e^2}} \\ \frac{-m_e v_e \cos\varphi}{\sqrt{1-v_e^2}} \\ \frac{m_e v_e \sin\varphi}{\sqrt{1-v_e^2}} \\ 0 \end{bmatrix} + \begin{bmatrix} E_\nu \\ -E_\nu \cos\theta \\ -E_\nu \sin\theta \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{m_p}{\sqrt{1-v_p^2}} \\ \frac{m_p v_p}{\sqrt{1-v_p^2}} \\ 0 \\ 0 \end{bmatrix}$$

MASSES

$m_p = 938.272$ MeV

$m_n = 939.565$ MeV

$m_e = 0.511$ MeV

CONSERVE P_x

$$\frac{m_p v_p}{\sqrt{1-v_p^2}} = E_\nu \cos\theta + \frac{m_e v_e \cos\varphi}{\sqrt{1-v_e^2}}$$

CONSERVE P_y

$$E_\nu \sin\theta = \frac{m_e v_e \sin\varphi}{\sqrt{1-v_e^2}}$$

UNKNOWNS

v_p

E_ν

θ

CONSERVE E

$$m_n = \frac{m_e}{\sqrt{1-v_e^2}} + E_\nu + \frac{m_p}{\sqrt{1-v_p^2}}$$

3 EQUATIONS, 3 UNKNOWNS; CAN BE SOLVED

COULD

- DO A FEW PAGES OF ALGEBRA
- DO ALGEBRA WITH MATHEMATICA OR SIMILAR AND PLUG IN
- PLUG IN AND SOLVE NUMERICALLY

$\longrightarrow$ I DO #3

RESULT:

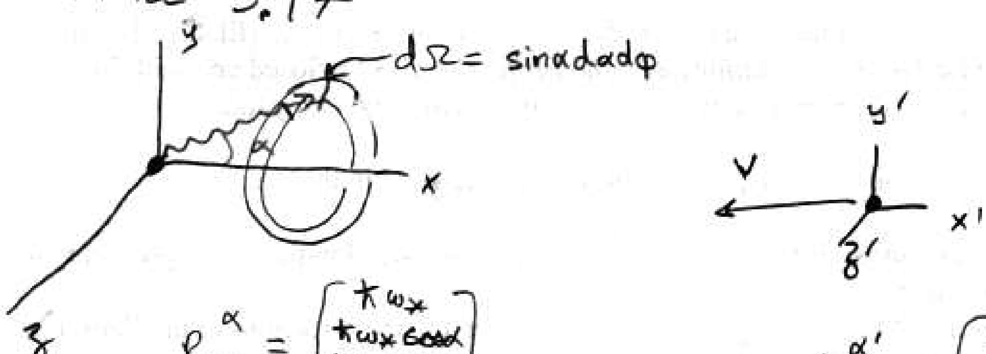
$$\boxed{\begin{aligned} v_p &= 0.00106 \\ E_\nu &= 0.520 \text{ MeV} \\ \theta &= 27.3^\circ \end{aligned}}$$

$$\text{SO } E_p = \frac{m_p}{\sqrt{1-v_p^2}} \approx m_p \dots$$

$$\underset{\sim}{P}_\nu = \begin{bmatrix} 0.520\,\text{MeV} \\ -0.462\,\text{MeV} \\ -0.238\,\text{MeV} \\ 0 \end{bmatrix}$$

2

MARTRE 5.17



$$p^{\alpha}_{phot} = \begin{bmatrix} \hbar\omega_* \\ \hbar\omega_* \cos\alpha \\ \hbar\omega_* \sin\alpha \\ 0 \end{bmatrix} \qquad \Lambda^{\alpha'}{}_{\beta} = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a)

$$p^{\alpha'}_{phot} = \begin{bmatrix} \hbar\omega_* \gamma (1 + v\cos\alpha) \\ \hbar\omega_* \gamma (v + \cos\alpha) \\ \hbar\omega_* \sin\alpha \\ 0 \end{bmatrix}$$

$$\cos\alpha' = \frac{p^x_{phot}}{p^t_{phot}}$$

$$\boxed{\cos\alpha' = \frac{v + \cos\alpha}{1 + v\cos\alpha}}$$

WHEE!

$\Rightarrow$

(b) $\frac{dN}{d\Omega dt}$ = CONST (ISOTROPIC)

$= \frac{dN}{\sin\alpha\, d\alpha\, d\phi\, dt}$

dN = # PHOTONS EMITTED INTO $d\Omega$ DURING dt

EMITTED

WE WANT $\frac{dN}{\sin\alpha' d\alpha' d\phi' dt'}$ OBSERVED

$d\phi' = d\phi$ SINCE THAT'S TRANSVERSE TO THE MOTION

$$\frac{dt'}{dt} = \gamma$$ TIME DILATION

*[ACTUALLY, THIS OMITS A FACTOR OF $1-v\cos\alpha'$ DUE TO THE "BUNCHING" OF PHOTONS AS THE SOURCE APPROACHES]

SINCE WE HAVE

$$\cos\alpha' = \frac{v+\cos\alpha}{1+v\cos\alpha}$$

WE ALSO HAVE

$$\cos\alpha = \frac{\cos\alpha' - v}{1-v\cos\alpha'}$$ [INVERSE TRANSFORM]

$$-\sin\alpha\, d\alpha = \frac{-\sin\alpha' d\alpha'}{1-v\cos\alpha'} - \frac{(\cos\alpha'-v)(+v\sin\alpha' d\alpha')}{(1-v\cos\alpha')^2}$$

$$\sin\alpha\, d\alpha = \sin\alpha' d\alpha' \left[\frac{1-v\cos\alpha' + v\cos\alpha' - v^2}{(1-v\cos\alpha')^2}\right]$$

$$\sin\alpha\, d\alpha = \frac{\sin\alpha' d\alpha'}{\gamma^2(1-v\cos\alpha')^2}$$

$$\frac{dN}{\sin\alpha\, d\alpha\, d\phi\, dt} = \frac{dN}{\sin\alpha' d\alpha' d\phi' dt'}\gamma^3(1-v\cos\alpha')^2$$

$$\boxed{\frac{dN}{d\Omega' dt'} = \gamma^{-3}(1-v\cos\alpha')^{-2}\frac{dN}{d\Omega dt}}$$

(c) NOW WE HAVE TO DEAL WITH THE FACT THAT EACH PHOTON IS REDSHIFTED

$$\frac{dE}{d\Omega dt} = \hbar\omega_* \frac{dN}{d\Omega dt}$$

SINCE EACH PHOTON HAS ENERGY $\hbar\omega_*$

WHAT IS $\frac{\omega'}{\omega_*}$?

$$k^{\alpha} = \begin{bmatrix} \hbar\omega_* \\ \hbar\omega_*\cos\alpha \\ \hbar\omega_*\sin\alpha \\ 0 \end{bmatrix}$$

$$k^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} k^{\beta} = \begin{bmatrix} \hbar\omega_*(\gamma + \gamma v\cos\alpha) \\ \hbar\omega_*(v + \gamma\cos\alpha) \\ \hbar\omega_*\sin\alpha \\ 0 \end{bmatrix}$$

$$\frac{\omega'}{\omega_*} = \gamma(\underbrace{1 + v\cos\alpha})$$

$$1 + v\cos\alpha = \frac{1 - v\cos\alpha'}{1 - v\cos\alpha'} + \frac{v\cos\alpha' - v^2}{1 - v\cos\alpha'}$$

$$= \frac{1 - v^2}{1 - v\cos\alpha'} = \gamma^{-2}(1 - v\cos\alpha')^{-1}$$

SO $\frac{\omega'}{\omega_*} = \gamma^{-1}(1 - v\cos\alpha')^{-1}$

THUS

$$\frac{dE}{d\Omega dt} = \gamma(1 - v\cos\alpha')\,\hbar\omega'\frac{dN}{d\Omega dt} = \gamma^4(1 - v\cos\alpha')^3\underbrace{\left(\hbar\omega'\frac{dN}{d\Omega' dt'}\right)}_{\frac{dE'}{d\Omega' dt'}}$$

$$\boxed{\frac{dE'}{d\Omega' dt'} = \gamma^{-4}(1 - v\cos\alpha')^{-3}\frac{dE}{d\Omega dt}}$$

⇐ BUT SEE * ON P.3 ⇒ SHOULD HAVE ANOTHER FACTOR OF $(1 - v\cos\alpha')^{-1}$

(d) SEE CLASS NOTES 2/22

3 HARTLE 5.22

ROCKET FRAME

MOMENTUM: $$0 = \frac{dm\,u}{\sqrt{1-u^2}} - \frac{(m-dm)\,dv}{\sqrt{1-dv^2}}$$

$$= \frac{dm\,u}{\sqrt{1-u^2}} - (m\,dv - \cancel{dm\,dv})(1 - \tfrac{1}{2}\cancel{dv^2})$$ BINOMIAL

KILL DIFFERENTIAL^2 TERMS AS DINKY.

$$0 = \frac{dm\,u}{\sqrt{1-u^2}} - m\,dv$$

$$dv\,\frac{\sqrt{1-u^2}}{u} = \frac{dm}{m}$$

$\boxed{\text{BUT}}$ WE WANT dv', WHICH IS THE dv IN THE GALAXY FRAME,

$\frac{dv}{dv'}$ WILL DEPEND ON v!

GALAXY FRAME IS GOING LEFT @ V, SO

$$\Lambda^{\alpha'}{}_{\beta} = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

GALAXY FRAME

$\Rightarrow$

FINAL ROCKET $\underset{\sim}{P}$:

ROCKET FRAME

GALAXY FRAME

$$P^{\alpha} = \begin{bmatrix} \frac{m-dm}{\sqrt{1-dv^2}} \\ \frac{(m-dm)dv}{\sqrt{1-dv^2}} \\ 0 \\ 0 \end{bmatrix} \qquad P^{\alpha'} = \begin{bmatrix} \frac{(m-dm)}{\sqrt{1-(v+dv')^2}} \\ \frac{(m-dm)(v+dv')}{\sqrt{1-(v+dv')^2}} \\ 0 \\ 0 \end{bmatrix}$$

$$\Lambda^{\alpha'}{}_{\beta} P^{\alpha} = \begin{bmatrix} \frac{(m-dm)(1+Vdv)}{\sqrt{(1-\cancel{dv^2})(1-v^2)}} \\ \frac{(m-dm)(V+dv)}{\sqrt{(1-\cancel{dv^2})(1-v^2)}} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{m-dm}{\sqrt{1-v^2-2Vdv'}} \\ \frac{mV+mdv'-Vdm}{\sqrt{1-v^2-2Vdv'}} \\ 0 \\ 0 \end{bmatrix}$$

THREW OUT $(dv')^2$'s

BINOMIALIZE & THROW OUT d^2 TERMS

$$\begin{bmatrix} \frac{(m-dm+mVdv)}{\sqrt{1-v^2}} \\ \frac{mV+mdv-Vdm}{\sqrt{1-v^2}} \end{bmatrix} = \begin{bmatrix} \frac{(m-dm)}{\sqrt{1-v^2}\sqrt{1-\frac{2vdv'}{1-v^2}}} \\ \frac{mV+mdv'-Vdm}{\sqrt{1-v^2}\sqrt{1-\frac{2vdv'}{1-v^2}}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{m-dm+mVdv}{\sqrt{1-v^2}} \\ \frac{mV+mdv-Vdm}{\sqrt{1-v^2}} \end{bmatrix} = \begin{bmatrix} \frac{(m-dm)\left(1+\frac{vdv'}{1-v^2}\right)}{\sqrt{1-v^2}} \\ \frac{(mV+mdv'-Vdm)\left(1+\frac{vdv'}{1-v^2}\right)}{\sqrt{1-v^2}} \end{bmatrix}$$

(YES, THIS WORKS; MULTIPLY IT OUT!)

$$\begin{bmatrix} \frac{m-dm+mVdv}{\sqrt{1-v^2}} \\ \frac{mV+mdv-Vdm}{\sqrt{1-v^2}} \end{bmatrix} = \begin{bmatrix} \frac{m-dm+m(\gamma^2 dv')}{\sqrt{1-v^2}} \\ \frac{mV+m(\gamma^2 dv')-Vdm}{\sqrt{1-v^2}} \end{bmatrix}$$

CLEARLY $\underline{\gamma^2 dv' = dV}$ $\Longrightarrow$

SO IN THE GALAXY FRAME, WE HAVE

$$\gamma^2 dv' \frac{\sqrt{1-u^2}}{u} = \frac{dm}{m}$$

WHAT I USED TO CALC dv'

$$\frac{\sqrt{1-u^2}}{u}\int_0^V \frac{dv}{(1-v^2)} = \int_{M_i}^{M_f} \frac{dm}{m} = \ln \frac{M_f}{M_i}$$

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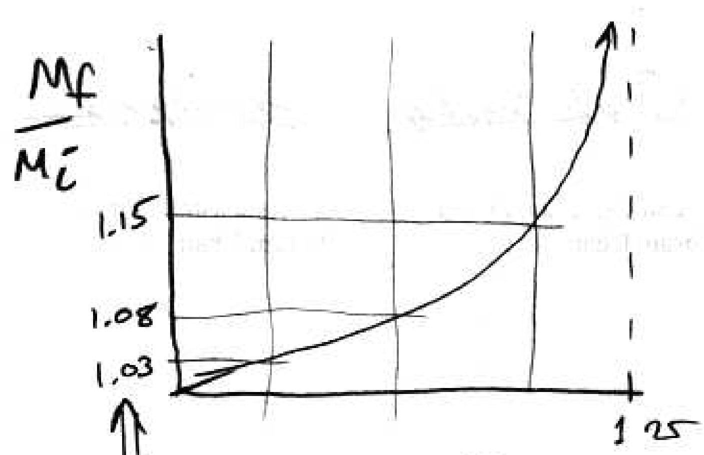
$$= \frac{\sqrt{1-u^2}}{u}\left[\frac{1}{2}\ln\frac{1+v}{1-v}\right]_0^V = \ln\frac{M_f}{M_i}$$

$$= \frac{\sqrt{1-u^2}}{u}\left[\frac{1}{2}\ln\left(\frac{1+V}{1-V}\right)\right] = \ln\frac{M_f}{M_i}$$

$$\boxed{\left(\frac{1+V}{1-V}\right)^{\sqrt{1-u^2}/2u} = \frac{M_f}{M_i}}$$



NUMBERS FOR $u=0.99$ (FAST EXHAUST)