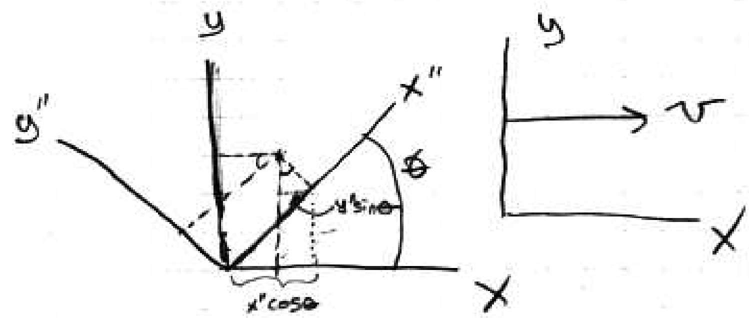
1



(a) $X = x''\cos\theta - y''\sin\theta$ $\Rightarrow$ $x'' = x\cos\theta + y\sin\theta$

$y = x''\sin\theta + y''\cos\theta$ $\qquad y'' = -x\sin\theta + y\cos\theta$

SO ROW

$$\boxed{R^{\alpha''}{}_{\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

COL

(b) THINK... THINK...

(c)

$$\Lambda^{\alpha''}{}_{\beta'} = R^{\alpha''}{}_{\gamma}\Lambda^{\gamma}{}_{\beta'} = \Lambda^{\gamma}{}_{\beta'} R^{\alpha''}{}_{\gamma}$$

THIS IS THE ORDERING FOR MATRIX MULT, BUT BOTH SUM EXPRESSIONS ARE RIGHT

(d) CAN DO MATRIX MULT HERE, IF WE ARE CAREFUL

$$\Lambda^{\alpha''}{}_{\beta'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ROW

$$\Lambda^{\alpha''}{}_{\beta'} = \begin{bmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v\cos\theta & \gamma\cos\theta & \sin\theta & 0 \\ -\gamma v\sin\theta & -\gamma\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

COLUMN

$\boxed{3}$

(a) $$u^{\alpha'} = \begin{bmatrix} 1/\sqrt{1-u^2} \\ 0 \\ u/\sqrt{1-u^2} \\ 0 \end{bmatrix}$$

(b) $$u^{\alpha''} = \Lambda^{\alpha''}{}_{\beta'} u^{\alpha'} = \begin{bmatrix} \Lambda^0{}_0 u^0 + \Lambda^0{}_2 u^2 \\ \Lambda^1{}_0 u^0 + \Lambda^1{}_2 u^2 \\ \Lambda^2{}_0 u^0 + \Lambda^2{}_2 u^2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\gamma}{\sqrt{1-u^2}} + 0 \\ \frac{\gamma v \cos\theta}{\sqrt{1-u^2}} + \frac{u \sin\theta}{\sqrt{1-u^2}} \\ \frac{-\gamma v \sin\theta}{\sqrt{1-u^2}} + \frac{u\cos\theta}{\sqrt{1-u^2}} \\ 0 \end{bmatrix}$$

$$\boxed{u^{\alpha''} = \frac{1}{\sqrt{1-u^2}} \begin{bmatrix} \gamma \\ \gamma v\cos\theta + u\sin\theta \\ -\gamma v \sin\theta + u\cos\theta \\ 0 \end{bmatrix}} \qquad \gamma \equiv \frac{1}{\sqrt{1-v^2}}$$

(c) $$g_{\alpha\beta} u^{\alpha''} u^{\beta''} = \frac{1}{1-u^2}\left\{ -\gamma^2 + (\gamma v \cos\theta + u\sin\theta)^2 + (-\gamma v \sin\theta + u\cos\theta)^2 \right\}$$

$$= \frac{1}{1-u^2}\Big\{ -\gamma^2 + \gamma^2 v^2 \cos^2\theta + u^2 \sin^2\theta + \cancel{\gamma v u \cos\theta \sin\theta}$$
$$+ \gamma^2 v^2 \sin^2\theta + u^2\cos^2\theta - \cancel{\gamma v u \cos\theta \sin\theta} \Big\}$$

$$= \frac{1}{1-u^2}\left\{ -\gamma^2 + \gamma^2 v^2 + u^2 \right\}$$

$\Rightarrow$

$$\underset{\sim}{u}\cdot\underset{\sim}{u} = \frac{1}{1-u^2}\left\{\frac{v^2-1}{1-v^2}+u^2\right\} = \frac{u^2-1}{1-u^2} = -1$$

WHEE!

3

(a)

$$300\,km = \int_{R_\oplus}^{R_\oplus+\Delta r} ds\Big|_{dt=0} \qquad ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1+\frac{2GM}{r}\right)dr^2$$

$$= \int_{R_\oplus}^{R_\oplus+\Delta r}\left(1+\frac{2GM}{r}\right)^{1/2}dr \approx \int_{R_\oplus}^{R_\oplus+\Delta r}\left(1+\frac{GM}{r}\right)dr$$

$$= \Delta r + GM\left[\ln\frac{R_\oplus+\Delta r}{R_\oplus}\right] \qquad \ln(1+x) \simeq x - \frac{x^2}{2}$$

$$300\,km \approx \Delta r + \frac{\Delta r}{R_\oplus}$$

$$\Delta r = 300\,km - \frac{\Delta r\, GM}{R_\oplus} \approx 300\,km - \frac{300\,km}{R_\oplus}GM$$

DIFF IN r COORD

$$\Delta r = 300\,km - \frac{300\,km}{6378\,km}\left(\frac{6.67\times10^{-11}\frac{m^3}{kg\,s^2}}{(3\times10^8\frac{m}{s})^2}\right)\cdot(5.972\times10^{24}\,kg)$$

$$\boxed{\Delta r - 300\,km = -2.1\times10^{-4}\,m}$$

THAT'S PRETTY SMALL!!!

(b)

$$d\tau = \sqrt{-ds^2} = \left(1-\frac{2GM}{R_\oplus}\right)^{1/2}dt \qquad \text{FOR CONST } r,\theta,\varphi$$

$$\frac{d\tau}{dt} = \left(1-\frac{2GM}{R_\oplus c^2}\right)^{1/2} \simeq 1 - \frac{GM}{R_\oplus c^2} = \boxed{1 - 7\times10^{-10}}$$

$\Longrightarrow$

(c) $$\frac{d\tau}{dt} = \left(1 - \frac{2GM_\odot}{(1AU)c^2}\right)^{1/2} \simeq 1 - \frac{GM_\odot}{(1AU)c^2}$$

$M_\odot = 2\times10^{30}$ kg $1\ AU = 1.5\times10^{11}$ m

$$\boxed{\frac{d\tau}{dt} = 1 - 10^{-8}}$$

SUN'S TIME DILATION IS (MUCH) MORE SIGNIFICANT, BUT EVEN THAT'S NOT SAYING MUCH....

4 HARTLE 6.6

NOTE: $\cosh^2\alpha - \sinh^2\alpha = 1$

(a) $$dt = \left(\frac{c}{g} + \frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right)\frac{g}{c}\,dt' + \frac{dx'}{c}\sinh\left(\frac{gt'}{c}\right)$$

$$dx = c\left(\frac{c}{g} + \frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right)\frac{g}{c}\,dt' + dx'\cosh\left(\frac{gt'}{c}\right)$$

$$c^2dt^2 = c^2(dt')^2\left[\left(\frac{c^2}{g^2} + \frac{2x'}{g} + \frac{(x')^2}{c^2}\right)\frac{g^2}{c^2}\cosh^2\left(\frac{gt'}{c}\right)\right] + (dx')^2\sinh^2\left(\frac{gt'}{c}\right)$$

$$dx^2 = c^2(dt')^2\left[\left(\frac{c^2}{g^2} + \frac{2x'}{g} + \frac{(x')^2}{c^2}\right)\frac{g^2}{c^2}\sinh^2\left(\frac{gt'}{c}\right)\right] + (dx')^2\cosh^2\left(\frac{gt'}{c}\right)$$

$$-c^2dt^2 + dx^2 = -c^2(dt')^2\left(\frac{c^2}{g^2} + \frac{2x'}{g} + \frac{(x')^2}{c^2}\right)\frac{g^2}{c^2} + (dx')^2$$

$$\boxed{ds^2 = -g^2(dt')^2\left(\frac{c}{g} + \frac{x'}{c}\right)^2 + (dx')^2} + (dy')^2 + (dz')^2$$

(b) FOR SOMETHING AT REST IN THIS FRAME, $dx' = 0$

THEN $$\frac{dx}{dt} = \frac{c\left(\frac{c}{g} + \frac{x'}{c}\right)\sinh\left(\frac{gt'}{c}\right)\frac{g}{c}dt'}{\left(\frac{c}{g} + \frac{x'}{c}\right)\cosh\left(\frac{gt'}{c}\right)\frac{g}{c}dt'}$$

$$\frac{dx}{dt} = c\,\frac{\sinh\left(\frac{gt'}{c}\right)}{\cosh\left(\frac{gt'}{c}\right)}$$

$$\sinh\alpha = \frac{e^{\alpha}-e^{-\alpha}}{2} \qquad \cosh\alpha = \frac{e^{\alpha}+e^{-\alpha}}{2}$$

FOR α SMALL:

$$\sinh\alpha \approx \frac{1+\alpha-1+\alpha}{2} \approx \alpha$$

$$\cosh\alpha \approx \frac{1+\alpha+1-\alpha}{2} \simeq 1$$

$$\boxed{\frac{dx}{dt} = gt'} \Longleftarrow$$

HEY, UNIFORM ACCEL

(IF $t' \approx t$, WHICH IT DOES IN THIS APPROX IF YOU SET $x'=0$)

C

$$\frac{d\tau\,(x'=h)}{d\tau\,(x'=0)} = \frac{+g\,(dt')\left(\frac{c}{g}+\frac{h}{c}\right)}{+g\,(dt')\left(\frac{c}{g}\right)}$$

$$\boxed{\frac{d\tau(x'=h)}{d\tau(x'=0)} = 1+\frac{gh}{c^2}}$$

JUST LIKE STATIC WEAK-FIELD FOR UNIFORM GRAV FIELD

$\Phi = gx$

!!!

$\Rightarrow$

5 HARTLE 6.13

S.W.F. METRIC w/ $\Phi = gz$: $ds^2 = -\left(1+\frac{2gz}{c^2}\right)(c\,dt)^2 + \left(1-\frac{2gz}{c^2}\right)dz^2$
$(dx = dy = 0)$

OBSERVER 2: $dz = 0$
$z = 0$ ⇐ CHOOSE

$$cT = c\tau_2 = \int_0^t \sqrt{-ds^2} = \int_0^t c\,dt = ct$$

T: TIME ELAPSED FOR $\mathcal{O}_2$

t: COORD @ END OF EXPERIMENT

$$T = \tau_2 = t$$

YAY!
THIS IS THE REASON FOR THE $z=0$ CHOICE; MAKES COORDINATES SIMPLER

$\mathcal{O}_1$: UP: $z = v_0 t - \frac{1}{2} g t^2$

DOWN: $z = h - \frac{1}{2} g t^2$

UP: $h = v_0\left(\frac{T}{2}\right) - \frac{1}{2} g \left(\frac{T}{2}\right)^2$ ⇒ REACHES $z = h$ @ $t = \frac{T}{2}$

$$h = \frac{v_0 T}{2} - \frac{gT^2}{8}$$

DOWN: $0 = h - \frac{1}{2} g \left(\frac{T}{2}\right)^2$

$$= h - \frac{gT^2}{8} \rightarrow h = \frac{gT^2}{8}$$

$$\frac{gT^2}{8} = \frac{v_0 T}{2} - \frac{gT^2}{8}$$

$$\frac{gT^2}{4} = \frac{v_0 T}{2}$$

$$v_0 = \frac{gT}{2}$$

NO SURPRISE...
$v = v_0 - gt$
$v = 0$ @ $t = \frac{T}{2}$ ⇒

WAY UP: $z = \frac{gT}{2}t - \frac{1}{2}gt^2$

$$dz = \left(\frac{gT}{2} - gt\right)dt \qquad dz = \left(\frac{gT}{2c} - \frac{gt}{c}\right)c\,dt$$

$$dz^2 = \left(\frac{g^2T^2}{4c^2} - \frac{g^2tT}{c^2} + \frac{g^2t^2}{c^2}\right)c^2dt^2$$

$$\frac{c\tau_1}{2} = \int_0^{T/2}\sqrt{-ds^2} = \int_0^{T/2}\left[\left(1+\frac{2gz}{c^2}\right)c^2dt^2 - \left(1-\frac{2gz}{c^2}\right)dz^2\right]^{1/2}$$

$$\frac{c\tau_1}{2} = \int_0^{T/2} c\,dt\left[\left(1+\frac{g^2Tt}{c^2} - \frac{g^2t^2}{c^2}\right) - \left(1-\frac{g^2Tt}{c^2}+\frac{g^2t^2}{c^2}\right)\left(\frac{g^2T^2}{4c^2} - \frac{g^2tT}{c^2} + \frac{g^2t^2}{c^2}\right)\right]^{1/2}$$

$$\frac{\tau_1}{2} = \int_0^{T/2} dt\left[1 + \frac{g^2Tt}{c^2} - \frac{g^2t^2}{c^2} - \frac{g^2T^2}{4c^2} + \frac{g^2tT}{c^2} - \frac{g^2t^2}{c^2} + O\left(\frac{1}{c^4}\right)\right]^{1/2}$$

$$= \int_0^{T/2} dt\left[1 + \frac{2g^2Tt}{c^2} - \frac{2g^2t^2}{c^2} - \frac{g^2T^2}{4c^2}\right]^{1/2} \approx \int_0^{T/2} dt\left[1 + \frac{g^2Tt}{c^2} - \frac{g^2t^2}{c^2} - \frac{g^2T^2}{8c^2}\right]$$

$$= \left[t + \frac{g^2Tt^2}{2c^2} - \frac{g^2t^3}{3c^2} - \frac{g^2T^2t}{8c^2}\right]_0^{T/2} = \frac{T}{2} + \frac{g^2T^3}{8c^2} - \frac{g^2T^3}{24c^2} - \frac{g^2T^3}{16c^2}$$

$$\tau_1 = T\left(1 + \frac{g^2T^2}{4c^2} - \frac{g^2T^2}{12c^2} - \frac{g^2T^2}{8c^2}\right) = \boxed{T\left(1 + \frac{g^2T^2}{24c^2}\right)}$$ ⇐ FREELY FALLING CLOCK

OBSERVER 3: $z = vt$ ← SAME h

WAY UP $h = v\frac{T}{2} = \frac{gT^2}{8}$

$$v = \frac{gT}{4}$$

$$z = \frac{gTt}{4} \qquad dz = \frac{gT}{4}dt$$

$$\frac{c\tau_3}{2} = \int_0^{T/2}\sqrt{-ds^2} = \int_0^{T/2}\left[\left(1+\frac{2gz}{c^2}\right)c^2dt^2 - \left(1-\frac{2gz}{c^2}\right)dz^2\right]^{1/2}$$

→

$$\frac{c\tau_3}{2} = \int_0^{T/2} c\,dt \left[\left(1+\frac{g^2Tt}{2c^2}\right) - \left(1-\frac{g^2Tt}{2c^2}\right)\frac{g^2T^2}{16c^2}\right]^{1/2}$$

$$= \int_0^{T/2} c\,dt \left[1+\frac{g^2Tt}{2c^2} - \frac{g^2T^2}{16c^2} + \mathcal{O}\left(\frac{1}{c^4}\right)\right]^{1/2}$$

$$\frac{\tau_3}{2} \cong \int_0^{T/2} dt \left[1+\frac{g^2Tt}{4c^2} - \frac{g^2T^2}{32c^2}\right] = \left[t + \frac{g^2Tt^2}{8c^2} - \frac{g^2T^2t}{32c^2}\right]_0^{T/2}$$

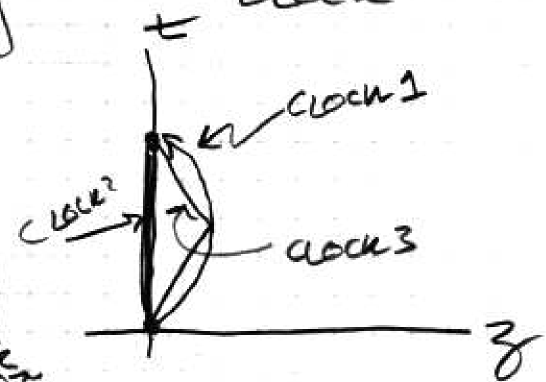
$$\frac{\tau_3}{2} = \frac{T}{2} + \frac{g^2T^3}{32c^2} - \frac{g^2T^3}{64c^2} = \frac{T}{2}\left(1+\frac{g^2T^2}{32c^2}\right)$$

$$\boxed{\tau_3 = T\left(1+\frac{g^2T^2}{32c^2}\right)}$$ $\Leftarrow$ LINEAR MOTION CLOCK

$$\tau_1 > \tau_3 > \tau_2$$

$\Uparrow$ FREE-FALLING CLOCK, EXPECT EXTREMAL PROPER TIME!

CLOSER PATH TO τ_1 THAN τ_2 IS



IF MOTION IS HORIZONTAL: $z=0$, $ds^2 = -dt^2 + dz^2 + dx^2$

ANALYSIS WILL LOOK JUST LIKE SR, SO CLOCK $\underline{\underline{2}}$ (UNACCELERATED) WILL HAVE MAX PROPER TIME $\Rightarrow$ THIS LOOKS JUST LIKE THE TWIN PARADOX !