1 (a) WE KNOW $\ell = r^2 \sin^2\cancel{\varphi} \frac{d\varphi}{d\tau}$

$$u^\varphi = \frac{d\varphi}{d\tau} = \frac{d\varphi}{dt}\frac{dt}{d\tau} = \Omega u^t$$

WE ALSO KNOW $\ell = \sqrt{12}\,M$ AT THE ISCO

$$\sqrt{12}\,M = (6M)^2 \underbrace{\cancel{\sin^2\theta}}_{1 \text{ FOR } \theta=\pi/2} \Omega u^t$$

$$\Omega = \frac{\sqrt{12}}{36M\, u^t}$$

WHAT IS u^t?

$$g_{\alpha\beta} u^\alpha u^\beta = -1 = g_{tt} u^t u^t + g_{\varphi\varphi} u^\varphi u^\varphi$$

$$-1 = -\left(1 - \frac{2M}{6M}\right)(u^t)^2 + (6M)^2 \Omega^2 (u^t)^2$$

$$= -\frac{2}{3}(u^t)^2 + \frac{(6M)^2 12}{6^4 M^2 \cancel{(u^t)^2}} \cancel{(u^t)^2}$$

$$-1 = -\frac{2}{3}(u^t)^2 + \frac{12}{36}$$

$$-1 = -\frac{2}{3}(u^t)^2 + \frac{1}{3}$$

$$-\frac{4}{3} = -\frac{2}{3}(u^t)^2$$

$$\underline{u^t = \sqrt{2}} \quad \leftarrow \text{PART OF (b)}$$

SO

$$\boxed{\Omega = \frac{\sqrt{6}}{36M}} \qquad \left(\Omega^2 = \frac{1}{6^3 M^2} = \frac{M^2}{r^3} \text{ AS IN 9.46}\right)$$

$$= \frac{1}{6^{3/2} M}$$

(b) SOLVED ABOVE

$$\boxed{u^\alpha = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \\ \frac{\sqrt{12}}{36M} \end{bmatrix}} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \\ \frac{1}{3\sqrt{12}M} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \\ \frac{1}{6\sqrt{3}M} \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \\ 0 \\ \frac{\sqrt{3}}{18M} \end{bmatrix}$$

$\Longrightarrow$

(c) $\Omega = \frac{d\varphi}{dt} = \frac{\sqrt{6}}{36M}$

$d\varphi = \frac{\sqrt{6}}{36M} dt$

$2\pi = \frac{\sqrt{6}}{36M} t$

$\boxed{t = \frac{36M(2\pi)}{\sqrt{6}}} = 12\sqrt{6}\pi M = 6^{3/2}(2\pi)M$

UNITS

$2.998 \times 10^8\ m = 1\ s$

$6.673 \times 10^{-11} \frac{m^3}{s^2} = 1\ kg$

SO

$2.476 \times 10^{-36}\ s = 1\ kg$

SUN: $M_\odot = 2.00 \times 10^{30}\ kg$

$M_\odot = 5.0 \times 10^{-6}\ s$

FOR $M = M_\odot$

$t = 0.0005\ sec \Rightarrow$ SPEEDY-ASS ORBIT!

(d) WE HAVE $\frac{dt}{d\tau} = \sqrt{2}$ (THAT'S u^t)

SO $d\tau = \frac{dt}{\sqrt{2}}$

$\boxed{\tau = \frac{36M(2\pi)}{\sqrt{12}}}$ FOR $M = M_\odot$, $\tau = 0.0003\ sec$

YIPERS

$= 6\sqrt{12}\pi M = 12\sqrt{3}\pi M$

$\boxed{2}$

(a) $\hbar\omega_1 = -g_{\alpha\beta} p^\alpha u^\beta$

SINCE $p^\varphi = 0$ & $u^r = 0$, ONLY g_{tt} MATTERS

$g_{tt} = -\left(1 - \frac{2M}{6M}\right) = -\frac{2}{3}$

$g_{\varphi\varphi} = (6M)^2$

$g_{rr} = \left(1 - \frac{2M}{6M}\right)^{-1} = \frac{3}{2}$

$\hbar\omega_1 = \frac{2}{3} p^t \sqrt{2}$

$\boxed{p^t = \frac{3\hbar\omega_1}{2\sqrt{2}}}$

NORMALIZE: $g_{\alpha\beta} p^\alpha p^\beta = 0$

RADIAL PHOTON: $p^\theta = p^\varphi = 0$

$-\frac{2}{3}\left(\frac{3\hbar\omega_1}{2\sqrt{2}}\right)^2 + \frac{3}{2}(p^r)^2 = 0$

$(p^r)^2 = \left(\frac{4}{9}\right)\left(\frac{9}{8}\right)(\hbar\omega_1)^2$

$\boxed{p^r = \frac{\hbar\omega_1}{\sqrt{2}}}$

(b) $\underset{\sim}{\xi} \cdot \underset{\sim}{p} = g_{tt}\xi^t p^t = \left(-\frac{2}{3}\right)(1)\left(\frac{3\hbar\omega_1}{2\sqrt{2}}\right)$

$$\boxed{\underset{\sim}{\xi} \cdot \underset{\sim}{p} = \frac{-\hbar\omega_1}{\sqrt{2}}}$$

(c) HERE, $\underset{\sim}{\xi} \cdot \underset{\sim}{p} = g_{tt}\xi^t p^t = (-1)(1)p^t$

$$-p^t = -\frac{\hbar\omega_1}{\sqrt{2}}$$

$$p^t = \frac{\hbar\omega_1}{\sqrt{2}}$$

SO $\boxed{\underset{\sim}{p} = \begin{bmatrix} \hbar\omega_1/\sqrt{2} \\ \hbar\omega_1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}}$

(d) $\boxed{\omega_2 = \frac{\omega_1}{\sqrt{2}}}$ $\left(-\underset{\sim}{u}_2 \cdot \underset{\sim}{p} = +p^t \quad \text{SINCE } g_{tt} = -1\right)$

(e) FROM CLASS: IF Θ_1 WERE FIXED IN SPACE REL. TO B.H.

$$\frac{\omega_2}{\omega_1} = \sqrt{1 - \frac{2M}{R_1}} = \sqrt{\frac{2}{3}} > \sqrt{\frac{1}{2}}$$

$\boxed{\omega_2 \text{ IS SMALLER} \Rightarrow \underline{\text{MORE}} \text{ REDSHIFT WHEN } \Theta_1 \text{ ORBITS}}$

(f) GREEN $\lambda \approx 5500\,\text{Å}$

DETECTED $\lambda = \sqrt{2}\,(5500\,\text{Å}) = \underline{\underline{7800\,\text{Å}}}$ VERY RED, MAYBE JUST BEYOND HUMAN VISION

λ GOES UP $\sqrt{2}$ IF ω GOES DOWN $\sqrt{2}$

3

(a) $$p^\alpha = \begin{bmatrix} m \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) $$\underset{\sim}{\xi} \cdot \underset{\sim}{p} = g_{tt} \xi^t p^t = \underline{\underline{-m}}$$

(c)

$$\underset{\sim}{\xi} \cdot \underset{\sim}{p} = -m = g_{tt} \xi^t p^t$$

@ $r=6M$

$g_{tt} = -\left(1-\frac{2M}{6M}\right) = -\frac{2}{3}$

$g_{rr} = \left(1-\frac{2M}{6M}\right)^{-1} = \frac{3}{2}$

$$-m = -\left(1-\frac{2M}{r}\right)(1)\, p^t$$

$$-m = -\frac{2}{3} p^t$$

$$\underline{\underline{p^t = \frac{3}{2} m}}$$

For p^r, NORMALIZE:

$$-m^2 = g_{tt}(p^t)^2 + g_{rr}(p^r)^2$$

$$p^r = \sqrt{\frac{-m^2 - g_{tt}(p^t)^2}{g_{rr}}}$$

$$p^r = m\sqrt{\frac{-1-\left(-\frac{2}{3}\right)\left(\frac{3}{2}\right)^2}{3/2}}$$

$$= m\sqrt{\frac{\frac{3}{2}-1}{\frac{3}{2}}}$$

$$= m\sqrt{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)}$$

$$p^r = \frac{m}{\sqrt{3}}$$

$$\boxed{p^\alpha = m\begin{bmatrix} 3/2 \\ 1/\sqrt{3} \\ 0 \\ 0 \end{bmatrix}}$$

(d) $E = -\underset{\sim}{u}_1 \cdot \underset{\sim}{p}$ SINCE $p^\phi = 0$ & $u_1^r = 0$, ONLY t MATTERS

FROM 1, WE HAVE $u^t = \sqrt{2}$ $\quad g_{tt} = -\frac{2}{3}$

$$E = +\frac{2}{3}\sqrt{2}\, p^t = \frac{2}{3}\sqrt{2}\,\frac{3}{2} m = \boxed{\sqrt{2}\, m = E}$$

(e) m ! THE PARTICLE IS AT REST WRT $\mathcal{O}_1$ NOW

(f) $\boxed{\Delta E = (\sqrt{2}-1)\, m}$

14 WE HAVE $E_{phot} = \hbar\omega_1 = (\sqrt{2}-1)\, m$

(a)

FROM 7(a), $p^t = \frac{3\, \hbar\omega_1}{2\sqrt{2}} = \frac{3(\sqrt{2}-1)}{2\sqrt{2}} m = p^t$

$p^\varphi = \frac{\sqrt{3}}{12M} \hbar\omega_1 = \frac{\sqrt{3}(\sqrt{2}-1)}{12M} m = p^\varphi$

(b)

$\underset{\sim}{\xi} \cdot \underset{\sim}{p}_{phot} = \text{const} = g_{tt}\, \xi^t\, p^t_{phot}$

$= \left(-\frac{2}{3}\right)(1)\left(\frac{3(\sqrt{2}-1)}{2\sqrt{2}} m\right) = -\frac{\sqrt{2}-1}{\sqrt{2}} m$

@ ∞, $\underset{\sim}{\xi} \cdot \underset{\sim}{p}_{phot} = (-1)(1)\, p^t_{phot}$

$p^t_{phot} = +\frac{\sqrt{2}-1}{\sqrt{2}} m$ $\qquad E_{phot} = -\underset{\sim}{u}_\infty \cdot \underset{\sim}{p}_{phot}$

SO $\boxed{E_{phot} = \frac{\sqrt{2}-1}{\sqrt{2}} m \text{ AT } \infty}$

$= 0.3\, m$

WAY BETTER THAN FUSION

(c) $kT \sim \Delta E = (-1+\sqrt{2})\, m$

$T \sim \frac{(-1+\sqrt{2})\, m}{k}$

(d) FOR $m = 1.67\times10^{-27}$ kg $\qquad T \sim \frac{(-1+\sqrt{2})(1.67\times10^{-27}\text{ kg})(3\times10^8 \frac{m}{s})^2}{1.38\times10^{-23}\text{ J/K}}$

$T \sim 3\times10^{12}$ K $\quad$ THAT'S HOT!

PEAK λ IS GAMMA RAY

e) CONT'D THAT'S **REALLY** HOT

WE'VE OVER SIMPLIFIED, OF COURSE ← AND THIS IS JUST THE INNER EDGE OF THE DISK...

$T \sim 10^5$-100'S OF MILLIONS **ARE** OBSERVED SOMETIMES,

AND SOME MORE DETAILED THEORY SUGGESTS INNER DISK T OF 10^{11} ISH K

f) $\sqrt{2}$ LOWER ⇒ STILL A BADASS GAMMA RAY!