A260 HW#5

1 HARTLE 18.6

"REST"-FRAME ← SN

WE KNOW $\Delta t = (1+z)\Delta\tau$

↑ OBSERVER FRAME

$\Delta t = 2$ MONTHS

$$\Delta\tau = \frac{2\text{ MONTHS}}{1+1.1} = \boxed{0.95\text{ MONTHS}}$$

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SR: $$\omega' = \omega\frac{\sqrt{1-v^2}}{1-V\cos\alpha'}$$

FOR RADIAL MOTION, $\cos\alpha' = 1$

$$\omega' = \omega\frac{\sqrt{1-v^2}}{1-V} = \omega\sqrt{\frac{1-v}{1+v}}$$

$$\lambda' = \lambda\sqrt{\frac{1+v}{1-v}}$$ (SINCE $\lambda\nu = 1$, $\nu = 2\pi\omega$)

$$z \equiv \frac{\lambda'-\lambda}{\lambda} = \sqrt{\frac{1+v}{1-v}} - 1$$

OR IT'S EASY ENOUGH TO DERIVE THIS FROM $\underset{\sim}{P} \propto \underset{\sim}{\Lambda}^{\alpha'}{}_{\beta}$ (DONE IN CLASS, I THINK)

$$1+z = \frac{1+v}{\sqrt{1-v^2}} = \gamma(1+v)$$

$$\frac{\Delta t}{\Delta\tau} = \gamma = \boxed{\frac{1+z}{1+v} > \frac{\Delta t}{\Delta\tau}}$$

NOT THE SAME THING!

$$(1+z)^2 = \frac{1+v}{1-v}$$

$$(1+z)^2 - v(1+z)^2 = 1+v$$

$$(1+z)^2 - 1 = v(1+(1+z)^2)$$

$$V = \frac{(1+z)^2-1}{(1+z)^2+1}$$

⇒

$$\frac{\Delta t}{\Delta \tau} = \frac{1+z}{1+\frac{(1+z)^2-1}{(1+z)^2+1}} = \frac{(1+z)\left[(1+z)^2+1\right]}{(1+z)^2+1+(1+z)^2-1}$$

$$\boxed{\frac{\Delta t}{\Delta \tau} = \frac{(1+z)\left[(1+z)^2+1\right]}{2(1+z)^2}}$$

COMPARE TO $\frac{\Delta t}{\Delta \tau} = 1+z$ FOR COSMOLOGY

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(a)

$$L = \frac{dE}{dt'} = \hbar\omega' \frac{dN}{dt'}$$

$$\boxed{\frac{dN}{dt'} = \frac{L}{\hbar\omega'}}$$

(b)

$$ds^2 = -dt^2 + a^2(t)\left[dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right]$$

$$A = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} a^2 r^2 d\theta \sin\theta\, d\phi$$

$$= a^2 r^2 \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \sin\theta\, d\theta\, d\phi$$

SINCE r, t HELD CONSTANT, $a = a(t)$

$$\boxed{A = 4\pi a^2 r^2}$$

EVALUATE A AT TIME OF DETECTION

(c)

$$\frac{dt}{dt'} = (1+z) = \frac{a_0}{a(t)}$$

TIME DILATION

$a(t) \leftarrow$ TIME OF EMISSION

$$\left(\frac{dN}{dA\,dt'} = \frac{L}{\hbar\omega' 4\pi \underbrace{a^2(t)}_{=1} r^2}\right)\left(\frac{dt'}{dt}\right)$$

$=1 \leftarrow a(t)$ @ TIME OF DETECTION (SET $a_0 = 1$ FOR CONVENIENCE)

$$\boxed{\frac{dN}{dA\,dt} = \frac{L\, a(t)}{\hbar\omega' 4\pi r^2}}$$

FOR $a_0 = 1$

(d)

$$\frac{dE}{dA\,dt} = \hbar\omega \frac{dN}{dA\,dt} = \boxed{\frac{\omega L\, a(t)}{\omega' 4\pi r^2} = \frac{dE}{dA\,dt}}$$

$\uparrow$ DETECTED FREQUENCY

$\Rightarrow$

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(B) WE KNOW $\frac{\omega}{\omega'} = \frac{1}{(1+z)}$ $\frac{1}{a(t)} = 1+z$ FOR $a_0 = 1$

SO

$$\frac{\omega}{\omega'} = \frac{a(t)}{a_0}$$

ACTUALLY, THIS ONE IS MORE FUNDAMENTAL

SO

$$\underbrace{\frac{dE}{dA\,dt}}_{\text{OBSERVED FLUX}} = \frac{L a^2(t)}{4\pi r^2} = \boxed{\frac{L}{4\pi r^2 (1+z)^2} = \frac{dE}{dA\,dt}}$$

IN CLASS, WE FOUND $r(z; H_0, \Omega_M, \Omega_\Lambda)$

RECAP: $\dot{a}^2 - \frac{8\pi\rho}{3} a^2 = 0$

$$\rho_0 = \underbrace{\frac{3 H_0^2}{8\pi}}_{\rho_c} (\Omega_M + \Omega_{vac})$$

DENSITY TODAY

$\Omega_M + \Omega_{vac} = 1$

$$\rho = \frac{3 H_0^2}{8\pi}\left(\frac{\Omega_M}{a^3} + \Omega_{vac}\right) \quad \text{FOR } a_0 = 1$$

$$\left(\rho_M = \frac{\rho_{M0}}{a^3}\right)$$

$$\dot{a}^2 = \frac{8\pi}{3} a^2 \frac{3 H_0^2}{8\pi}\left(\frac{\Omega_M}{a^3} + \Omega_{vac}\right)$$

$$\dot{a}^2 = H_0^2\left(\frac{\Omega_M}{a} + \Omega_{vac} a^2\right)$$

$$1+z = \frac{a_0}{a}$$

$$\dot{z} = -\frac{a_0}{a^2}\dot{a} = -\frac{\dot{a}}{a^2} \quad \text{FOR } a_0 = 1$$

$$\dot{a}^2 = a^4 \dot{z}^2 = (1+z)^{-4} \dot{z}^2$$

$$(1+z)^{-4} \dot{z}^2 = H_0^2\left(\Omega_M (1+z) + \frac{\Omega_{vac}}{(1+z)^2}\right)$$

$$\dot{z}^2 = H_0^2\left(\Omega_M (1+z)^5 + \Omega_{vac} (1+z)^2\right)$$

$$\dot{z} = (1+z) H_0 \sqrt{\Omega_M (1+z)^3 + \Omega_{vac}}$$

$\Rightarrow$

$$\frac{dz}{dt} = (1+z)\, H_0 \sqrt{\Omega_M (1+z)^3 + \Omega_{vac}}$$

CONSIDER PATH OF PHOTON FROM SOURCE TO OBSERVER:

$$ds^2 = 0 = -dt^2 + a^2(t)\, dr^2$$

$$\frac{dr}{dt} = \frac{1}{a(t)} = (1+z)$$

$$\frac{dz}{dr} = H_0 \sqrt{\Omega_M (1+z)^3 + \Omega_{vac}}$$

$$r = \int_0^z \frac{dz}{H_0 \left(\Omega_M (1+z)^3 + \Omega_{vac}\right)^{1/2}}$$

WE DID THIS IN GROUP PROBLEMS

SO

$$\boxed{\frac{dE}{dA\,dt} = \frac{L}{4\pi (1+z)^2 \left(\int \frac{dz}{H_0 \left(\Omega_M (1+z)^3 + \Omega_{vac}\right)^{1/2}} \right)^2}}$$

WRITE $\frac{dE}{dA\,dt} = \frac{L}{4\pi d_L^2}$

$$d_L = (1+z)\, r = (1+z) \int_0^z \frac{dz}{H_0 \left(\Omega_M (1+z)^3 + \Omega_{vac}\right)^{1/2}}$$

"LUMINOSITY DISTANCE"

COORDINATE DISTANCE

CARROLL: $H_0 d_L = (1+z) \int_0^z \frac{dz}{\left[(1+z)^2 \underbrace{(1+\Omega_M z)}_{\Omega_M (1+z) - \Omega_M + 1 \,=\, \Omega_M (1+z) + \Omega_{vac}} - z(2+z)\Omega_\Lambda \right]^{1/2}}$

$$= (1+z) \int_0^z \frac{dz}{\left[(1+z)^3 \Omega_M + \Omega_\Lambda (1+z)^2 - (1+z)^2 \Omega_\Lambda + \Omega_\Lambda \right]^{1/2}}$$

$$= (1+z) \int_0^z \frac{dz}{\left[(1+z)^3 \Omega_M + \Omega_\Lambda \right]^{1/2}}$$

SAME THING

4 (a) GENERAL EQUATION:

$$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = -k$$

FOR $k=0$

$$\dot{a}^2 = \frac{8\pi\rho}{3}a^2$$

$$\boxed{\rho_c = \frac{3}{8\pi}\frac{\dot{a}^2}{a^2}}$$

TODAY, $\rho_{c0} = \frac{3 H_0^2}{8\pi}$

SINCE $H_0 = \left(\frac{\dot{a}}{a}\right)_{t=\text{NOW}}$

(b)

$$\frac{\rho}{\rho_c} = \frac{8\pi}{3}\frac{a^2}{\dot{a}^2}\rho = \frac{8\pi a^2 \rho}{3\left(\frac{8\pi\rho}{3}a^2 - k\right)}$$

$$\frac{d}{dt}\left(\frac{\rho}{\rho_c}\right) = \frac{8\pi a^2 \dot{\rho}}{3\left(\frac{8\pi\rho}{3}a^2 - k\right)} - \frac{8\pi a^2 \rho}{3\left(\frac{8\pi\rho}{3}a^2 - k\right)^2}\left[\frac{8\pi\dot{\rho}}{3}a^2 + \frac{16\pi\rho a\dot{a}}{3}\right]$$

$$+ \frac{16\pi\rho a\dot{a}}{3\left(\frac{8\pi\rho}{3}a^2 - k\right)}$$

$$= \frac{8\pi}{3\left(\frac{8\pi\rho}{3}a^2 - k\right)^2}\left[a^2\dot{\rho}\left(\frac{8\pi\rho}{\cancel{3}}a^2 - k\right) - \frac{8\pi a^4 \rho\dot{\rho}}{\cancel{3}} - \frac{16\pi a^3 \dot{a}\rho^2}{\cancel{3}} + 2\rho a\dot{a}\left(\frac{8\pi\rho}{\cancel{3}}a^2 - k\right)\right]$$

$$\frac{d}{dt}\left(\frac{\rho}{\rho_c}\right) = \frac{8\pi}{3\left(\frac{4\pi\rho}{3}a^2 - k\right)^2}\left[-ka^2\dot{\rho} - 2ka\rho\dot{a}\right]$$

$= 0$ IF $k=0$ $(\rho = \rho_c)$

WHEE!

⟹

SUPPOSE EXPANDING UNIVERSE:

$\dot{a} > 0 \qquad \ddot{a} < 0$ ← DECEL

$\dot{\rho} \leq 0$ ← $=0$ IF IT'S ALL VACUUM, <0 IF MATTER $\rho \propto \frac{1}{a^3}$, <0 IF MIX

$$\frac{d}{dt}\left(\frac{\rho}{\rho_c}\right) = \underbrace{\frac{8\pi}{3(\;)^2}}_{\text{THIS IS POSITIVE SINCE IT'S SQUARED}}(-k)\underbrace{\left[2a\dot{a}\rho + a^2\dot{\rho}\right]}$$

FRIEDMANN EQUATION

$$\dot{a}^2 - \frac{8\pi\rho}{3}a^2 = -k$$

$$2\dot{a}\ddot{a} - \frac{8\pi}{3}a^2\dot{\rho} - \frac{16\pi}{3}\rho a\dot{a} = 0$$

$$\frac{6\dot{a}\ddot{a}}{8\pi} = \underbrace{a^2\dot{\rho} + 2a\dot{a}\rho}_{\text{HEY! FAMILIAR!}}$$

WHY AM I DOING THIS? THE PROBLEM SUGGESTS THAT $\ddot{a}<0$ MATTERS, SO I'M LOOKING FOR $\ddot{a}$.

$$\frac{d}{dt}\left(\frac{\rho}{\rho_c}\right) = \frac{8\pi}{3(\;)^2}(-k)\underbrace{\left[\frac{6\dot{a}\ddot{a}}{8\pi}\right]}_{<0 \text{ FOR } \dot{a}>0,\ \ddot{a}<0}$$

IF $\rho < \rho_c$, OPEN, $k = -1$,

$\frac{d}{dt}\left(\frac{\rho}{\rho_c}\right) < 0 \quad \therefore \frac{d}{dt}\Omega_{tot} < 0$ IF $\Omega_{tot} < 1$

IF $\rho > \rho_c$, CLOSED, $k = +1$

$\frac{d}{dt}\left(\frac{\rho}{\rho_c}\right) > 0 \quad \therefore \frac{d}{dt}\Omega_{tot} > 0$ IF $\Omega_{tot} > 1$

THUS ρ DIVERGES FROM ρ_c IN A DECELERATING NON-FLAT UNIVERSE!!!