1

(a) DO WHAT I SAY.

(b) DO THAT

HEY, IT'D WORK.

I CALL THIS THEORY "INDIGNANT DESIGN"

2

OBSERVER @ $r = R_1$

$$g_{\alpha\beta} = \begin{bmatrix} -(1-\frac{2M}{r}) & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

$$\underline{u}\cdot\underline{u} = -1$$

$$= g_{\alpha\beta}u^\alpha u^\beta$$

$u^r = u^\theta = u^\phi = 0$

FOR $\frac{dr}{d\tau} = 0$

$\frac{d\theta}{d\tau} = 0$

$\frac{d\phi}{d\tau} = 0$

$$-1 = -\left(1-\frac{2M}{r}\right)(u^0)^2$$

$$u^0 = \left(1-\frac{2M}{R_1}\right)^{-1/2}$$

$$\hbar\omega_1 = -\underline{u}\cdot\underline{p}$$

$$= -g_{\alpha\beta}u^\alpha p^\beta$$

$$= -g_{00}u^0p^0 - g_{01}u^0p^1$$

'CAUSE $u^1 = u^2 = u^3 = 0$ & $p^2 = p^3 = 0$ FOR RADIAL PHOTON

$$\hbar\omega_1 = +\left(1-\frac{2M}{R_1}\right)\left(1-\frac{2M}{R_1}\right)^{-1/2}p^0 - (1)\left(1-\frac{2M}{R_1}\right)^{-1/2}p^1$$

ALSO

$$\underline{p}\cdot\underline{p} = 0$$

$$g_{\alpha\beta}p^\alpha p^\beta = g_{00}p^0p^0 + g_{01}p^0p^1 + g_{10}p^1p^0 = 0$$

$$0 = -\left(1-\frac{2M}{R_1}\right)(p^0)^2 + (1)\,p^0p^1 + (1)\,p^0p^1$$

$$\frac{1}{2}\left(1-\frac{2M}{R_1}\right)p^0 = p^1$$

$$\hbar\omega_1 = \left(1-\frac{2M}{R_1}\right)^{1/2}p^0 - \left(1-\frac{2M}{R_1}\right)^{-1/2}\left(1-\frac{2M}{R_1}\right)\frac{p^0}{2}$$

$\Rightarrow$

$\boxed{2}$ CON'T'D

$$\hbar\omega_1 = \frac{1}{2}\left(1-\frac{2M}{R_1}\right)^{1/2} p^0$$

$$p^0 = \frac{2\hbar\omega_1}{\left(1-\frac{2M}{R_1}\right)^{1/2}}$$

$$p^1 = \left(1-\frac{2M}{R_1}\right)^{1/2} \hbar\omega_1$$

WHEN THE PHOTON IS @ $r=R_1$

($p^1 > 0$ FOR OUTWARD PHOTON)

FOR THE PHOTON (WHICH IS FREELY FALLING),

$$\underline{\xi}\cdot\underline{p} = \text{const}$$

IF WE REPLACE $v \to v + v_0$ FOR CONST v_0, ds^2 IS UNCHANGED. THUS, WE HAVE A KILLING VECTOR

$$\xi = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \leftarrow v \\ \leftarrow r \\ \leftarrow \theta \\ \leftarrow \varphi \end{array}$$

$$\underline{\xi}\cdot\underline{p} = g_{\alpha\beta}\,\xi^\alpha p^\beta$$

$$= g_{00}(\xi^0)(p^0) + g_{01}(\xi^0)(p^1)$$

ALL OTHER TERMS DON'T CONTRIBUTE

$$= -\left(1-\frac{2M}{R_1}\right)(1)\frac{2\hbar\omega_1}{\left(1-\frac{2M}{R_1}\right)^{1/2}} + (1)(1)\left(1-\frac{2M}{R_1}\right)^{1/2}\hbar\omega_1$$

$$\boxed{\underline{\xi}\cdot\underline{p} = -\hbar\omega_1\left(1-\frac{2M}{R_1}\right)^{1/2}}$$

CONSERVED ALONG PHOTON'S MOTION

$\boxed{2}$ CONT'D

OBSERVER 2

$$\underline{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{FOR } R_2 \gg 2M$$

$$\underline{u} \cdot \underline{p} = -\hbar\omega_2$$

WHAT IS $\underline{p}$?

TWO EQUATIONS:

$$\underline{\xi} \cdot \underline{p} = -\hbar\omega_1 \left(1 - \frac{2M}{R_1}\right)^{1/2}$$

$$\underline{p} \cdot \underline{p} = 0$$

FROM SYMMETRY, WE KNOW
$p^\theta = 0,\ p^\phi = 0$
SINCE THE PHOTON STARTED RADIAL

As $r \to \infty$, $g_{\alpha\beta} \to \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$

$$\underline{p} \cdot \underline{p} = g_{\alpha\beta} p^\alpha p^\beta = g_{00}(p^0)^2 + g_{01}(p^0)(p^1) + g_{10}(p^1)(p^0)$$

$$= -(p^0)^2 + 2p^0p^1$$

$$\frac{p^0}{2} = p_1$$

$$\underline{\xi} \cdot \underline{p} = g_{00}\xi^0 p^0 + g_{01}\xi^0 p^1 = -\hbar\omega_1 \left(1 - \frac{2M}{R_1}\right)^{1/2}$$

$$= -1(1)(p^0) + (1)(1)p^1$$

$$= -p^0 + \frac{p^0}{2} = -\hbar\omega_1 \left(1 - \frac{2M}{R_1}\right)^{1/2}$$

$$p^0 = 2\hbar\omega_1 \left(1 - \frac{2M}{R_1}\right)^{1/2}$$

$$p_1 = \hbar\omega_1 \left(1 - \frac{2M}{R_1}\right)^{1/2}$$

2 CONT'D

$$\underset{\uparrow}{\underline{u}} \cdot \underline{p} = -\hbar\omega_2 = +g_{00}u^0p^0 + g_{01}u^0p^1$$

FOR $\mathcal{O}_2$

$$-\hbar\omega_2 = (-1)(2\hbar\omega_1)\left(1-\frac{2M}{R_1}\right)^{1/2} + (1)(1)(\hbar\omega_1)\left(1-\frac{2M}{R_1}\right)^{1/2}$$

$$= -\hbar\omega_1\left(1-\frac{2M}{R_1}\right)^{1/2}$$

$$\frac{\omega_2}{\omega_1} = \left(1-\frac{2M}{R_1}\right)^{1/2}$$

YAY!!!!!

3

THE KEY HERE IS THAT SINCE THESE ARE <u>VECTOR</u> EQUATIONS, IF THEY WORK IN ONE FRAME THEY WORK IN <u>ALL</u> FRAMES.

SO WORK IN THE MOST CONVENIENT FRAME → THE MCRF OF THE OBSERVER

$$u^\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad g_{\alpha\beta} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

THEN

(a)

$$\left[(p\cdot u)^2 + p\cdot p\right]^{1/2}$$

$$= \left[(g_{\alpha\beta}p^\alpha u^\beta) + g_{\alpha\beta}p^\alpha p^\beta\right]^{1/2}$$

$$= \left[(-1p^0u^0)^2 + (-1(p^0)^2 + (p^1)^2 + (p^2)^2 + (p^3)^2\right]$$

$$= (p^1)^2 + (p^2)^2 + (p^3)^2 = |\vec{p}| \checkmark$$

(b)

$$\frac{\underline{p} + (\underline{p}\cdot\underline{u})\underline{u}}{-\underline{p}\cdot\underline{u}}$$

FOR MCRF, $\underline{p}\cdot\underline{u} = -E = -\gamma m c^2$

$\uparrow$ OF PARTICLE

$$\frac{\underline{p} + (-E)\underline{u}}{+E} = \frac{\underline{p}}{E} - \underline{u}$$

$$= \frac{\underline{p}}{E} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{p} = \begin{bmatrix} \gamma m c^2 \\ \gamma m v^x \\ \gamma m v^y \\ \gamma m v^z \end{bmatrix}$$

$$\frac{\underline{p}}{E} = \begin{bmatrix} 1 \\ v^x \\ v^y \\ v^z \end{bmatrix} \quad \{\text{FOR } c=1)$$

$$\frac{\underline{p}}{E} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v^x \\ v^y \\ v^z \end{bmatrix} \checkmark$$

YAY!