Astronomy 253 — Group Problems 6

Friday, March 19

1. In this problem, you will work through the justification of the "divergence theorem". This theorem in turn leads to Poisson's equation, which is a (relatively) simple relationship between the density and gravitational potential in a Galaxy.

   Consider a tiny (i.e. differentially sized) cubical volume of space which is h on each side. Suppose in the space in and around this volume there is some vector field $\vec{f}$. (If you want to make it concrete, consider the vector field to be the gravitational field of surrounding stars.) This vector field may be changing with position.

   Choose the coordinates (x_0, y_0, z_0) to be the lower-left corner of this cubical volume.

   (a) Draw a little picture to indicate that we all know what we're talking about.

   (b) Write down an expression for the divergence the vector field $\vec{f}$ in terms of the the value of the field on the corners of the box, h, and unit vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$.

   (c) Consider $\vec{dS}$, a differential surface element. Each side of the cubical volume we're considering is a different $\vec{dS}$. $\vec{dS}$ is defined so that its magnitude is the area of the surface element (so that it has units length squared), and it's direction is perpendicular to the surface element in the *outward* direction. (I.e. $\vec{dS}$ points directly away from the box on all six sides of our cubical volume element.)

   Write down six expressions: one each for $\vec{f} \cdot \vec{dS}$ along each surface, in terms of $\vec{f}$ at each surface, h, and the coordinates x_0, y_0, and z_0.

   Sum these six expressions together to get $\vec{f} \cdot \vec{dS}$ over the whole surface of the box. How does this compare to the answer you had for (b)? Define a variable dV, the volume of our cube, and write down an equality between $\vec{\nabla} \cdot \vec{f}$ and $\vec{f} \cdot \vec{dS}$, using dV and $\vec{dS}$ in your expression but *not* using h.

   (d) Now consider a second cubical volume displaced along the x-axis from the first by a distance h. How does $\vec{f} \cdot \vec{dS}$ on the *lower* x side of this volume compare to $\vec{f} \cdot \vec{dS}$ on the *higher* x side of the original volume?

   If you want to calculate the sum of $\vec{f} \cdot \vec{dS}$ over all surfaces on *both* of these boxes, which surfaces must you consider?

   (e) Now consider stacking together many, many little boxes in all directions to make one ginormous arbitrary volume in space. Put together your answers to (c) and (d) to write down an expression relating $\vec{f}$ at the surface of this volume to $\vec{\nabla} \cdot \vec{f}$ within the volume.

   Assuming you did this properly, what you have written down is an expression of the *Divergence Theorem.*

2. From the definition of the gravitational potential, in class we had:

$$\frac{\vec{F}(\vec{r})}{-m} = \vec{\nabla}\Phi = \iiint \frac{G\,\rho(\vec{r}\,')\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} d^3r\,'$$

In this problem, you will take the divergence of both sides. On the left, you will get $\nabla^2\Phi$, which is defined as $\vec{\nabla}\cdot\vec{\nabla}\Phi$, and is an operated called the *Laplacian*. On the right, you will get...

(a) Argue why you can move the $\vec{\nabla}\cdot$ operator inside the integral, and that once it's there, you can move the $G\,\rho(r\,')$ *outside* of the derivative. (Remember that a divergence is just a derivative!)

(b) Write out the remaining terms inside the derivative (which should be all $\vec{r}$ and $\vec{r}\,'$) in terms of Cartesian coordinates (x, y, z) and (x', y', z').

(c) Take a deep breath.

(d) Crank out the derivative that you're left within part (b). What does this derivative come down to? Why does the analysis that you've done not apply for $\vec{r} = \vec{r}\,'$?

(e) Decide not to worry about the case where $\vec{r} = \vec{r}\,'$ for now, but use your result from (d) to argue that the value of the right side of the expression is not changed if you alter the limits of integration to a tiny sphere centered around $\vec{r}\,' = \vec{r}$

(f) If we're only considering $\vec{r}\,' \simeq \vec{r}$, then $\rho(\vec{r}\,') \simeq \rho(\vec{r})$. For a differential volume, this equality should be very good. Once you've written it as $\rho(\vec{r})$, it no longer depends on the variables of integration. Pull that puppy out of the integral.

(g) With what's left, how must you change the expression if you want to replace the $\vec{\nabla}\cdot$ with $\vec{\nabla}'\cdot$? (That is, what else must you change if you want to take the derivatives in terms of x', y', and z' instead of in terms of x, y, and z.)

(h) Use the divergence theorem to get rid of that derivative.

3. If you worked through the previous problem correctly, you should have ended up with an expression:

$$\nabla^2\Phi \;=\; -G\,\rho(\vec{r}) \oiint \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3}\cdot d\vec{S}$$

Where the right side indicates a surface integral over the entire (closed) surface of the volume you are considering (which, remember, is now a tiny sphere centered around $\vec{r}\,' = \vec{r}$).

(a) OK, you can let that deep breath out now.

(b) Change variables so that $\vec{R}$=$\vec{r}\,' - \vec{r}$. This sounds all complicated, but really it makes things much easier. $\vec{R}$ is now a vector that points from the center of the little spherical volume you're considering, which is centered around the point of interest (where you were trying to find the potential in the first place).

(c) Write down an expression for $d\vec{S}$ on the surface of our little spherical volume in terms of spherical coordinates R, θ, and ϕ, and $\hat{R}$, the unit vector in the R direction.

(d) Substitute this into your expression. Notice what happened to all of the R's! Be happy that you no longer have to worry about all the dividing-by-zero that would have given you trouble in problem 2e.

(e) No, really, be happy.

(f) Evaluate the integrals. What you're left with is *Poisson's Equation.* This relates the density at any point to the second derivative of the gravitational potential at that point.