A311, 2005-11-11

1. **(a)** From linearizing the mass and momentum equations (neglecting viscosity and gravitational potential), we showed that small perturbations propogate as waves with speed:

$$a_{s0} = \sqrt{\frac{k\,T_0\,\gamma}{m}}$$

Show that in the case of an isentropic gas ($P\rho^{-\gamma} = \text{const}$) that this is also equal to:

$$a_{s0}^2 = \frac{\partial P}{\partial \rho}.$$

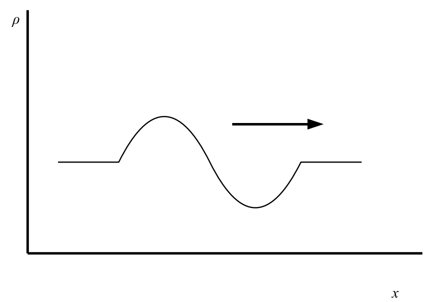
(b) Go back and repeat the derviation of the wave equations. This time, however, don't put in the equation of state for P, but just write the most general expression (i.e. the chain rule) for converting $\vec{\nabla} P$ to $\vec{\nabla}\rho$. Show that the equation you get out is:

$$\frac{\partial^2 \delta\rho}{\partial t^2} - \frac{\partial P}{\partial \rho}\nabla^2 \delta\rho = 0$$

This is, of course, the 3d wave equation with wave speed $a_s = \sqrt{\frac{\partial P}{\partial \rho}}$

(c) Now you're going to do something very scary: conclude something about the *non-linear* behavior of the equations from their linearized solution... However, the qualitative result that you will come up with matches what you get when you do a nonlinear solution.

Consider a small wave propogating through a fluid in the $+x$ direction:



How does the sound speed at the *peak* of the wave (where $\rho > \rho_0$) compare to the sound speed at the *trough* of the wave (where $\rho < \rho_0$)? What does this tell you about how the shape of the wave, and in particular the slope of density gradients, will change as the wave propogates?

2. Consider a uniformly expanding Universe. Pick a spot, and go into the frame of reference where the gas at that spot is at rest. Never look too far from that spot, so that you don't really have to worry about relativity.

Describe the expanding Universe by introducing a "scale factor" $R(t)$, so that the distance between two points is $d = R(t)d_0$, d_0 being some "initial" distance when $R(t) = 1$. (Set $R = 1$ at $t = 0$ if you like.)

(a) (This one is supposed to be easy.) If the Universe is homogeneous and isotropic, what is $\nabla\dot{\rho}$?

(b) What is $\nabla \cdot \vec{u}$, in terms of $R(t)$, in this situation?

(c) What is $\frac{\partial \rho}{\partial t}$, in terms of $R(t)$? (Don't figure this out from the continuity equation; just think about what density *is*, and our definition of the scale factor.)

(d) Show that (a), (b), and (c) together are consistent with the mass continuity equation.

You may find it helpful to know that in spherical coordinates *with spherical symmetry*, the gradient and divergence can be written:

$$\vec{\nabla} A = \frac{\partial A}{\partial r}$$

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\,V_r\right)$$