Chapter 1

Units and Dimensionality

If somebody asked me how tall I am, I might respond "1.78". But what do I mean by that? 1.78 feet? 1.78 miles? In fact, my height is 1.78 meters. Most physical measurements have *dimensionality* to them. That is, they are meaningless unless you attach a unit to them. *Dimensionality* means the type of unit. For example, inches, meters, miles, and light-years are all length units; something measured in those units have dimensionality of length. Kilograms, grams, and solar masses are all units of dimensionality mass. Measurements of different dimensionalities cannot be meaningfully compared. How many kilograms are there in a meter? The question does not even make sense.

There are some dimensionless quantities. For example, ratios are nearly always dimensionless. How many times older than my nephew am I? I am seven times older; that seven doesn't have any units on it, as it's a ratio of two ages (42 years and 6 years, respectively). For any other number you report, it's essential that you report the units of the number along with the number itself. Otherwise, you haven't completely specified what you're talking about.

1.1 SI Units

There is a "standard international" system of units. You may ask, why does anybody ever use anything other than these? SI Units are a good set of units for everyday measurements. However, they are very clumsy when dealing with the very small or the very large. When talking about atoms, or about stars, it's often convenient to use other units that are better matched to the scale of the system. What's more, some places historically use other units; for instance, the United States still uses the British Imperial system of units.

There are a finite number of dimensionalities. For purposes of this course, there are only four dimensionalities that you need to know about. They are, with their SI units, listed below:

Dimensionality	**SI Unit**
Length	m
Mass	kg
Time	s
Electric Charge	C

The four core dimensionalities are length, mass, time, and electric charge.[1] For each dimensionality there can be a lot of different units. Something of dimensionality length can be measured in *any* length unit, but cannot be measured with a (say) time unit. It doesn't make sense directly to compare quantities of different dimensionalities. So, I could measure my height in feet— 5.84 feet is my height— or in meters. While clearly the number 1.78 does not equal 5.84, 1.78 meters *does* equal 5.84 feet. A measurement with dimensionality is clearly different from a pure number; the units on the number affect what that number means.

You are already familiar with the meter, kilogram, and second. (Indeed, because of these three base units, the SI system is sometimes called the "MKS" system.) You may or may not have heard of the Coulomb before. All other units that we will deal with are derived from these base units. For instance, consider velocity. The *dimensionality* of velocity is length over time (sometimes written L/T). Any unit that corresponds to a length divided by a time is a valid velocity unit; that could be kilometers per hour, miles per hour, or furlongs per fortnight. The dimensionality of velocity is neither length nor time, but is composed of those two dimensionalities. The SI unit for velocity is meters per second, or m/s. Sometimes derived units have their own names. Below is a table of some of the more important derived units in the SI system:

Dimensionality	Unit	Definition
Force	Newtons N	$\mathrm{kg\,m\,s^{-2}}$
Energy	Joules J	$\mathrm{N\,m} = \mathrm{kg\,m^2\,s^{-2}}$
Power	Watts W	$\mathrm{J/s} = \mathrm{kg\,m^2\,s^{-3}}$

(Remember that something raised to the negative power is in the denominator. Thus, one Newton is "one kilogram times meter per second squared", or $\mathrm{kg\,m/s^2}$.) While we can say that "force" is the dimensionality of force— as in the table above— that

[1] In fact, in the SI system, electric current rather than electric charge is considered a core dimensionality. However, it's conceptually more simple to consider charge as the core unit, and current as a derived unit, so I'll use that in this document.

is exactly the same as saying it is a dimensionality of mass times length divided by time squared, or $M\,L/T^2$, or $M\,L\,T^{-2}$.

Some people will "just always work with SI units", and then not write down units to go with their numbers during intermediate calculations. The idea is that since you're always using the standard, the final result of any series of calculations will be in the SI unit for whatever it is that you calculated. Even though, if you are careful, you can get away with this, it would still be wise to write down the units that go with numbers every time you write down those numbers. There are two primary reasons for this. First, it makes it much clearer what you are doing and what these intermediate numbers actually are. Without that, anybody reading your calculations may have a hard time following them, and you have not communicated as effectively as you might have. Second, by keeping track of your units throughout your calculation, you provide yourself with a cross-check: does your final answer have the units that it's *supposed* to have? If it doesn't, then that's a sign that you've made a calculation mistake somewhere along the way.

For example, suppose I told you that the density of water is 1 gram per cubic centimeter, and I wanted you to tell me how much mass there is in a spherical drop of water with radius 0.2 cm. First, let's convert to SI units; if you do it right, you can figure out that 1 g/cm^3 equals 1000 kg/m^3. Also, 0.2 cm is equal to 0.002 m. If you say that the volume of a sphere is πr^2, you could calculate the volume from this number:

$$Vol \;=\; \pi\,(0.002)^2 \;=\; 1.257\times 10^{-5}$$

Then, multiply the volume by the density to get the mass:

$$m \;=\; (1000)(1.257\times 10^{-5}) \;=\; 0.013$$

Figuring that you've done everything in SI units, you should get an answer in the SI unit for mass, so you could write down and box $\boxed{m = 0.013\text{ kg}}$. However, **this answer is wrong**. Did you see where it went wrong? Let's redo the problem, this time keeping track of units:

$$\begin{aligned} m &= (Vol)(dens) \\ &= (\pi\,(0.002\,\mathrm{m})^2)\left(\frac{1\,\mathrm{kg}}{\mathrm{m}^3}\right) \\ &= (1.257\times 10^{-5}\,\mathrm{m}^2)\left(\frac{1\,\mathrm{kg}}{\mathrm{m}^3}\right) \\ &= 0.0127\,\frac{\mathrm{kg}}{\mathrm{m}} \end{aligned}$$

Notice in the last step we cancelled the meter^2 in the numerator with *two* of the three meters in the denominator's meter^3. But, wait! This doesn't leave us with an answer that has dimensionality mass, it has dimensionality mass per length! Clearly we've done something wrong. In this case, the mistake was in our formula for volume. In fact, the volume of a sphere is of radius r is $\frac{4}{3}\pi r^3$. We caught this error because, by keeping track of the units as we were putting numbers into the calculation, we saw that the units didn't work out right. If you put in the right formula for volume, you discover that there are only 3.4×10^{-5} kg of water in a droplet that's 2 mm in radius.

1.1.1 SI Prefixes

Some "derived" units are just a prefix in front of a regular unit. There is a standard set of SI prefixes that can be prepended to any unit in order to make another unit of the same dimensionality but of a different size. The ones you are probably most familiar with are milli and kilo. A millimeter is 1/1000 of a meter, and a kilometer is 1000 meters. You could do the same thing with seconds; a millisecond is 0.001 seconds, and a kilosecond is 1000 seconds (about 17 minutes). Indeed, the SI mass unit, the kilogram, is itself 1000 grams. In this class, we will frequently talk about things that are much smaller, such as nanometers and microseconds. If you are in an astronomy class, you might find yourself using the mega or giga prefixes more often. The table below summarizes the prefixes.

Prefix	**Abbreviation**	**Multiplier**
terra	T	10^{12}
giga	G	10^{9}
mega	M	10^{6}
kilo	k	10^{3}
deci	d	10^{-1}
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}
femto	f	10^{-15}

Notice that case matters. There is very big difference between a Mm and a mm—a factor of a billion, in fact! The letter used to indicate micro is the Greek letter mu. There are a million μs in one second. The prefixes deci and centi are not used very often, and generally only with meters. While you will talk about centimeters, nobody generally talks about centigrams or centiseconds.

1.2 Arithmetic with Dimensional Quantities

When you put together numbers that have dimensions on them, you have to keep track of the units as you are doing your arithmetic. You can do algebra with numbers that have dimensions on them. **However, it is not a good idea in general to do algebra with numbers.** Solve things symbolically first, and only put in the numbers at the end. When you do this, you will have various quantities with different units.

When adding or subtracting numbers with units, you need to make sure that they have the *same* units. First of all, it doesn't make sense to add numbers with different dimensionality. One meter plus one kilogram isn't even meaningful. One meter plus one inch *is* meaningful, but it is not equal to two anything. You need to convert one of the two units to the other before adding the numbers. You could write one meter as 39 inches, and then say that one meter plus one inch is equal to 40 inches.

Multiplying and dividing units is more interesting. In this case, you treat the units just as if they were algebraic variables. If you multiply meters by meters, you get meters squared (or m^2). If you divide seconds cubed (s^3) by seconds, you get seconds squared (s^2). If you raise a quantity with units to a power, you have to remember to raise every part of that quantity's units to the same power. For example, you may be calculating the kinetic energy of a car massing 1,500 kg moving at 20 meters per second:

$$KE = \frac{1}{2} m v^2 = \frac{1}{2}(1500\,\mathrm{kg})\left(20\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = \frac{1}{2}(1500\,\mathrm{kg})\left(400\,\frac{\mathrm{m}^2}{\mathrm{s}^2}\right) = 3.0\times10^5\,\frac{\mathrm{kg\,m}^2}{\mathrm{s}^2}$$

Notice that the squared on the velocity is applied to the number, to the meters, *and* to the seconds.

1.3 The Unit Factor Method

Sometimes you will need to convert one unit to another unit. The trick for doing this: **multiply by one as many times as necessary**. You can always multiply a number by 1 without changing that number. The secret is writing the number 1 in a particularly clever way. Here are some ways you can write the number 1:

$$1 = \left(\frac{60\ \mathrm{min}}{1\ \mathrm{hr}}\right)$$

$$1 = \left(\frac{2.54\ \mathrm{cm}}{1\ \mathrm{in}}\right)$$

$$1 = \left(\frac{1\mathrm{M}_\odot}{2.0\times10^{30}\ \mathrm{kg}}\right)$$

(The $M_\odot$ in the last example is the standard symbol for the mass of the Sun.)

If you have an expression in one set of units and you need them in another set of units, you just multiply by one as many times as necessary. Cancel out units that appear anywhere on *both* the top and bottom in your huge product, and you will be left with a number and another set of units. A simple example: convert the length 2.500 yards into centimeters:

$$2.500\text{ yd} = (2.5\text{ yd})\left(\frac{36\text{ in}}{1\text{ yd}}\right)\left(\frac{2.54\text{ cm}}{1\text{ in}}\right) = (2.5\times 36\times 2.54)\text{ cm} = 228.6\text{ cm}$$

Notice that yards (yd) appear in the numerator and the denominator, and so get canceled out, as does inches. We're left with just cm. All we did was multiply the value 2.5 yd by 1, so we didn't change it at all; 228.6 cm is another way of saying 2.500 yd.

Another example: suppose I tell you that the surface area of the Sun is 2.4×10^{19} square meters. How many square miles is that?

$$\left(2.4\times 10^{19}\text{ m}^2\right)\left(\frac{100\text{ cm}}{1\text{ m}}\right)^2\left(\frac{1\text{ in}}{2.54\text{ cm}}\right)^2\left(\frac{1\text{ ft}}{12\text{ in}}\right)^2\left(\frac{1\text{ mi}}{5280\text{ ft}}\right)^2$$

Two things to notice about this. First, notice how all the unit factors are *squared*. That's because we started with meters squared at the beginning, which is meters times meters. If we're going to get rid of both of them, we have to divide by meters twice. The same then goes for all of the other units. Next, notice that everything except for the left-over miles squared cancel out. We're left with a bunch of numbers we can punch into our calculator (remembering to square things) to get:

$$\frac{(2.4\times 10^{19})(100^2)}{(2.54^2)(12^2)(5280^2)}\text{ mi}^2 = 9.3\times 10^{12}\text{ mi}^2$$

One more example. Sometimes you have more than one unit to convert. If I tell you that a car moves 60 miles per hour, how many meters per second is it going? (Notice here that instead of arduously multiplying out the conversion between meters and miles as I did in the previous example, I've looked up that there are about 1609 meters in one mile.)

$$\left(60\ \frac{\text{mi}}{\text{h}}\right)\left(\frac{1609\text{ m}}{1\text{ mi}}\right)\left(\frac{1\text{ h}}{60\text{ min}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right) = 27\ \frac{\text{m}}{\text{s}}$$

Note that since hours was originally in the *denominator*, we had to make sure to put it in the numerator in a later unit factor to make it go away (since we didn't want any hours in our final answer).

With this simple method, you can convert any quantity from one set of units to another set of units, keeping track of all the conversions as you do so.

1.4 Significant Figures

Suppose I tell you that one stick is 1.0 meters long, and that it is 4.7 times longer than another stick. How long is the second stick? Writing the words as equations (see previous section), you might write:

$$l_1 = 4.7\ l_2$$

l_1 is what you know (1.0 meters), and l_2 is what you're looking for, so solve the equation for l_2:

$$l_2 = \frac{l_1}{4.7}$$

plug in the numbers and solve for the answer:

$$l_2 = \frac{1.0\ \mathrm{m}}{4.7} = 0.212765957447\ \mathrm{m}$$

That answer is wrong! Why? Because it is expressed with too many significant figures.

Think about the original problem. I told you a stick was 1.0 meters long. Notice that I didn't say 1.00 meters long; only 1.0 meters long. That means that I was only willing to commit to knowing the length of the stick to within a tenth of a meter. It might really be more like 1.04 meters long, or perhaps 0.98 meters long, but I've rounded to the nearest tenth of a meter. Since I only know the length of the stick to about ten percent, and since I used that number to calculate the length of the second stick, I can't know the length of the second stick to the huge precision that I quote above— even though that is the "right" number that my calculator gave me. Given that I only know that the first stick is 1.0 meters long, and it is 4.7 times the length of the second stick, *all that I can say I know* about the length of the second stick is:

$$l_2 = 0.21\ \mathrm{m}$$

By saying this, I'm implicitly saying that I don't know the length of this second stick to better than the hundredths place. . . and I don't! Implicitly, I'm saying that I know the length of the second stick to about one part in 21. That's actually a bit better than I really know it (which is just to one part in 10, or to 10%, as that's all the better I know the length of the first stick), but this is the best you can do with just significant figures. (To do better, you have to keep track of not just units, but also uncertainties on every number. Doing so is an important part of the analysis of data in physics experiments. However, propagating uncertainties is beyond the scope of this course.)

How well you know a given number you write down is the reasoning behind significant figures. The basic idea is that you shouldn't report a number to more significant figures than you know are right. The rules can sometimes seem arbitrary, but if you think about them in terms of the basic idea behind them, they can start to make sense. There are four basic rules of significant figures:

1. When multiplying or dividing numbers, the answer has as many significant digits as that member of the product or quotient that has fewer significant digits. So, if I multiply 3.14159 by 2.0, the answer is 6.3; I round the answer to two significant figures, because 2.0 (the member of the product with fewer significant figures) only has two. This rule is an expression of the *percent uncertainty* in the figures that are going into your result. If you only know a number to within (say) 5%, then you will generally only have two significant figures on that number. You can't know the result of anything you multiply or divide by that number to better than 5% either, so the result won't have more significant figures than the number that went into it.

   Sometimes, it makes sense to report your result to one more or one less significant figure than what went into the calculation. This will make sense if you understand the "percent uncertainty" reasoning behind the number. For instance, if I tell you one stick 95 meters long, and another stick is exactly 1/9 as long as the first stick, the significant figure rule would suggest that you should only keep two figures, and report the answer as 11 meters long. However, the two significant figures on the first number means that you know it to about one part in 95. It would be better to report the answer as 10.6 meters long, since a result that is implicitly good to one part in 106 is much closer to your true precision than a result that is implicitly good only to one part in 11.

2. When adding or subtracting numbers, the answer is precise to the decimal place of the *least precise* member of the sum. If I add 10.02 meters to 2.3 meters, the answer is 12.3 meters. The second number was only good to the first decimal place, so the sum is only good to the first decimal place. Notice that the *number* of significant figures here is different from either number that went into the sum. When *multiplying*, it is the *number* of significant figures that is important; when *adding*, it is the *decimal place* that is important.

   Note that if I were to add 10.02 meters to 2.30 meters, the answer would be 12.32 meters; in this case, both members of the sum are significant to the hundreds place. It *is* possible to gain significant figures doing this. If you add 6.34 meters to 8.21 meters, each significant to three figures, the result is 14.55 meters, now significant to four figures.

   This rule makes sense again if you remember that significant figures represent the precision of a number. To what decimal place do you know all the things that you are adding or subtracting? You can't know the result to better than that decimal place.

3. A number which is *exact* should not go into considerations of significant figures. For example, suppose you're doing a unit factor conversion, and you multiply by the factor (12 in/1 ft). Your answer need not be limited to two significant figures because of this; there are *exactly* 12 inches in one foot. That's a *definition*; there is no uncertainty associated with it. In the first rule above, when I told you that the second stick was *exactly* 1/9 as long as the first stick, the 9 in 1/9 was a "perfect" number: you were told it was exact. Thus, that there is only one significant figure in the number 9 did *not* come into consideration for the number of significant figures in the answer.

4. **Always keep at least two or three more figures during intermediate calculations than you will report as significant figures in your final answer.** This is one of the two most common mistakes I observe in student work. (The other is thoughtlessly reporting your answer to however many digits your calculator gave you.) Otherwise, "round-off" errors will accumulate, and you may get the final answer wrong even though your general method and equations were correct. Consider, for example, summing the numbers 6.1 and 5.3, and multiplying the overall result by 4.1. The sum will be good to the first decimal place, and the final number will only be good to two significant figures because of the two significant figures in 4.1. The result is:

$$\begin{aligned} 5.3 + 6.1 &= 11.4 \\ (11.4)\,(4.1) &= 46.74 = 47 \text{ to two sig figs} \end{aligned}$$

   If, however, you round too soon, and don't keep the .4 at the end of the 11.4:

$$(11)\,(4.1) = 45.1 = 45 \text{ to two sig figs}$$

   In fact, you're now wrong! Even though both 11 and 4.1 are good to two significant figures, your result is incorrect to two significant figures. This is an example of "roundoff" error, where you lose precision by rounding numbers too soon.

You don't always have to get the number of significant figures *exactly* right. Significant figures are, after all, just an approximation of correctly taking into account and propagating your uncertainties, which is a topic that those who do more advanced studies in physical science will have to address. Just be reasonable, and make sure you understand the rationale behind why an answer might have a limited number of significant figures. It will often be acceptable to report an answer to one too many significant figures. However, it is technically incorrect to report a number that obviously has too many significant digits; in that case, you're misrepresenting your knowledge. By the same token, don't report a number with too *few* significant figures either, as in that case you're underselling what you know!

1.5 Dimensional Analysis

You can sometimes figure out something about a physical quantity just by considering its dimensionality. If you know what sorts of things *might* affect that quantity, and you have good reason to believe that it is just powers of those things multiplied together to give you that quantity, you may be able to figure out (up to a dimensionless constant) the equation that relates that quantity to the things that might affect it just by figuring out what makes the units work.

Consider the example of a simple pendulum: a small mass (the "bob") hangs at the end of a string. The other end of the string is fixed. The bob may oscillate back and forth. We want to figure out what is the equation for the period P (i.e. the length of time it takes to go through one oscillation). If we think about things that could affect that, there are three obvious possibilities. The first is the mass m of the bob at the end of the pendulum, the second is the length l of the string connecting the bob to the point from which the pendulum hangs, and the third is g, the acceleration due to gravity. For each of these quantities, we'll write down the dimensionality in terms of mass (M), length (L), and time (T). (Note that M here means mass, not meters!)

$$\begin{aligned}[m] &= M \\ [l] &= L \\ [P] &= T \\ [g] &= L/T^2\end{aligned}$$

The "bracket" notation, here, means "dimensionality of". So, the dimensionality of the period is time; the dimensionality of acceleration is length divided by time squared.

If the period is a product of various powers of the different quantities, then we can write:

$$[P] \;=\; [m]^a\,[l]^b\,[g]^c$$

The period itself wouldn't be equal to this, as there may well be (and, in fact, there is) a dimensionless quantity multiplying everything else. However, even if we don't get the right formula, we can figure out how the period depends on these other things.

Now, put in the dimensions for each quantity:

$$T \;=\; M^a\,L^b\left(\frac{L}{T^2}\right)^c$$

$$T \;=\; \frac{M^a\,L^{b+c}}{T^{2c}}$$

Matching up the powers of each dimensionality on the left— which is simple, there is only T to the first power— to the powers on the right, we get these three equations:

$$\begin{aligned} a &= 0 \\ b + c &= 0 \\ -2c &= 1 \end{aligned}$$

In this case, the equations are easy to solve. The bottom equation gives us $c = -1/2$, and that together with the second equation gives us $b = 1/2$. So, we now know that:

$$P \propto l^{1/2} g^{-1/2}$$

$$P \propto \sqrt{\frac{l}{g}}$$

Without doing any of the actual physics to figure out the period of the pendulum, but *only* by considering the units on each quantity, we've figured out that the period must be proportional to $\sqrt{l/g}$. (If you want to figure out the dimensionless constant in front of $\sqrt{l/g}$, then in fact you do need to consider the full physics.)