Chapter 3

Momentum and Angular Momentum

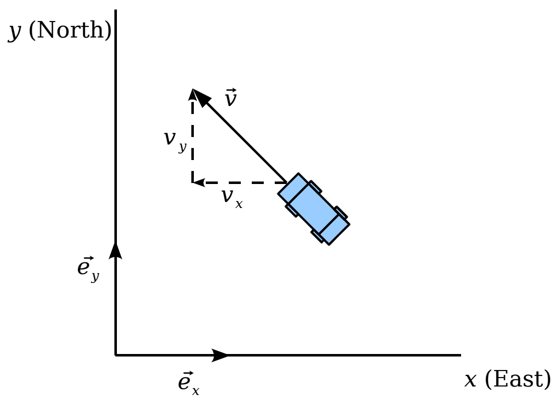
Chapter 2 introduced energy as a mathematical construct that has turned out to be very useful. There are two other conserved quantities that show up throughout all of our theories of physics. Both of those have to do with motion, but are different from kinetic energy. These other quantities are *momentum* and *angular momentum*.

3.1 Vectors

Both momentum and angular momentum are *vector* quantities. Later, we'll be talking about a more abstract form of vector used to represent the state of quantum systems. Here, we're talking about a special kind of vector, a vector in regular old 3d-space. Distinguish these from more general vectors, I shall call them 3-vectors, in reference to 3-dimensional space. A 3-vector is anything that has both a magnitude (size) and direction. For example, consider speed: the speed an object is moving is just a number. (We would call that a *scalar*.) Likewise, the kinetic energy of an object is a scalar; it's an amount of energy, and there is no direction associated with it. In contrast, the velocity of an object includes not just its speed, but also its direction. So, you could say that the speed of a car is 80 km/h. If you wanted to specify its velocity, you'd also have to give its direction. For example, you could say that the velocity of a car is 80 km/h due northwest.

3-vectors corresponding to different physical quantities will have different dimensionalites (and thus different units) associated with them. *Displacement* is a 3-vector form of distance. Distance just tells you how far apart two things are. Displacement tells you how far apart and in what direction. Just like distance, displacement comes in length units. So, you might say that one person is 1 meter due east of another person; in this case, the displacement from the other person to the first person is 1 m due east.. The *magnitude* of a 3-vector is just its size. Distance is the magnitude of displacement. If you consider that person whose displacement was 1 meter due east of the other person, you could also say that the distance between the two people was 1 meter. This is correct, even though it has less information.

A 3-vector can be visualized as an arrow in space. The length of the arrow represents the magnitude, and the direction the arrow points is the direction of the 3-vector. So, for example, let us consider a car going at 50 km/h due northwest.



The picture shows the x and y axes, representing East and North respectively. The z axis not drawn; it's up, straight out of the page. We notate vectors by drawing a little arrow on top of them; you can see the $\vec{v}$ in the diagram referring to the velocity of the car. Also shown are $\vec{e}_x$ and $\vec{e}_y$, the two *basis vectors*. In the case of 3-vectors, we can also call the basis vectors *unit vectors*, as they are 3-vectors whose length 1 (dimensionless), and that point right along the axes. The basis vectors define the coordinate system that we're using; here, they just define x and y for us. The car's velocity 3-vector $\vec{v}$ is represented by the direction and length of the arrow sticking out of the front of its picture. We can see that it points partially in the negative-x direction, and partly in the positive-y direction.

If you have a complete set of basis vectors, you can construct any other vector out of them. The three unit 3-vectors $\vec{e}_x$, $\vec{e}_y$, and $\vec{e}_z$, all 3-vectors of length dimensionless 1 pointing (respectively) along the x, y, and z axes, form the most obvious and most generally useful set of basis 3-vectors. Any other 3-vector can be written as a sum of constants times those basis vectors. So, here, we could say that:

$$\vec{v} \;=\; v_x\,\vec{e}_x \;+\; v_y\,\vec{e}_y \;+\; v_z\,\vec{e}_z$$

From looking at the picture, we can see that in this case v_x is going to have to be negative, v_y is going to have to be positive, and v_z is going to be zero. How do we figure out where they are? Well, we know that the car is going due northwest, so we expect that the absolute value of v_x and v_y will be the same (it's got just as much north velocity as west velocity). For the total speed, that is the total length of the 3-vector, we recognize that there's a right triangle there, and use a generalization of the Pythagorean Theorem:

$$v^2 = |\vec{v}|^2 = {v_x}^2 + {v_y}^2 + {v_z}^2$$

$$v = \sqrt{{v_x}^2 + {v_y}^2 + {v_z}^2}$$

In this example, we know that $v = 50$ km/h. For this to work, we have to have $v_x = -35$ km/h and $v_y = 35$ km/h.

3.2 Momentum

Kinetic energy is a quantity that's associated with motion. However, kinetic energy itself is not always conserved. If a cue pool ball runs into another ball, and the cue ball stops dead, the other ball goes off with the same speed that the cue ball came in at. In this case, the two balls have the same mass, so $\frac{1}{2}mv^2$ is the same both before and after the collision; kinetic energy *is* conserved in the collision. However, if two cars hit each other in a head-on collision, and the tangled wreck of the two cars stops dead at the point of impact, kinetic energy is clearly not conserved, as the v of everything after the collision is zero. It's not kinetic energy that's conserved, but total energy. The kinetic energy that the cars had before the collision is, during the collision, converted into other forms of energy: heat, noise, and possibly some potential energy as the structure of the car is rearranged. So, *sometimes*, in some collisions, kinetic energy is conserved. However, in other collisions, kinetic energy is not conserved. Note that total energy is always conserved; it's just that there are forms of energy other than kinetic energy, and sometimes kinetic energy can be converted to or from those other forms.

However, there is a quantity of motion that is conserved in every collision. If it is to be conserved in both the examples above, it can't just be based on the speeds of the particles. While the speed would seem to be enough in the example of the pool balls, in the example of the cars there was a lot of speed to start with, but *no* speed after the collision. To work in both of these examples (and in general), this conserved quantity has to be something that takes into account both speed *and* direction. That quantity is momentum. It is traditional to use the letter p to represent momentum. The momentum of a particle is defined by:

$$\vec{p} = m\vec{v}$$

where m is the mass of the particle, and $\vec{v}$ is the *velocity* of the particle. The magnitude of $\vec{v}$ is traditionally written $|\vec{v}|$, but is often just abbreviated as v without the arrow. Magnitudes of 3-vectors are *always* positive or zero; it does not make sense to say a 3-vector has a negative magnitude. The magnitude of velocity is what we call speed. You can't have a speed of -50 km/h, but you *can* be moving at 50 km/h in the negative-x direction.

One way of dealing with 3-vectors is to break them into components— an x-component v_x, a y-component v_y, and a z-component v_z. For now, to keep things simple, we'll only consider motion in one dimension, so that particles will not have any component of velocity in the y or z directions. Therefore, we can say that the particle's velocity is v_x in the $+x$ direction. If v_x is negative, it means that the particle is moving to the left. The *speed*, however, the magnitude of the velocity, is still positive; that's just how fast it's going, without reference to direction.

Just like total energy, it turns out that momentum is a conserved quantity. If you take everything into account (which is occasionally tricky), the total momentum before and after a collision or interaction must be the same. Consider the example of the two pool balls above. If the pool balls have a mass m and an x-velocity v_x, then the initial momentum is just mv_x. After the collision, it's the other ball that's moving, but the speed is the same, so the final momentum is mv_x. The total momentum is conserved.

In the case of the two cars colliding, suppose that both cars have the same mass m and are approaching each other with speed v. The car that is moving to the right has x-momentum $mv_{x1} = mv$, and the car that's moving to the left has x-momentum $mv_{x2} = -mv$. Notice that the x-momentum of the car moving to the left is *negative*! In contrast, the kinetic energy of both cars is positive, and is the same: $\frac{1}{2}mv^2$. The *total* momentum in the system is the sum of the momentum of the individual particles. Thus, the total x-momentum is $mv_x \; + \; m(-v_x) = 0$. After the collision, the velocity is zero, so the total momentum is still zero. Momentum is, in fact, conserved in the collision.

3.2.1 The Units of Momentum

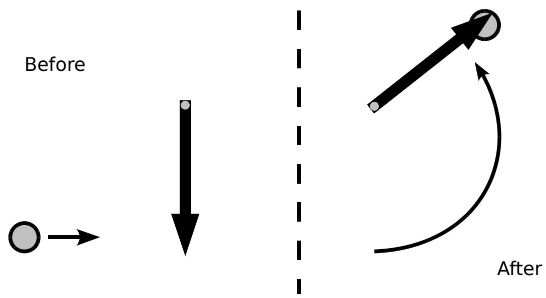
There isn't a special name for the units of momentum. If you look at the equation

$$\vec{p} \; = \; m\,\vec{v}$$

don't let the vector signs bother you. Velocity has dimensionality of length over time just like speed. If you multiply that by mass, you get a dimensionality of mass times length divided by time. Because the dimensionality must be the same on both sides of the equation, that is also the dimensionality of momentum. In the SI system, momentum comes in units of $\mathrm{kg\,m\,s^{-1}}$.

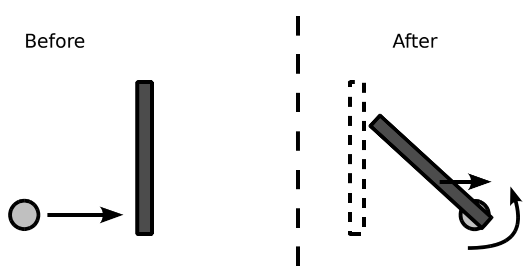
3.3 Angular momentum

Imagine the following experiment. You've got, somehow, a frictionless plane. (These frictionless planes are common in physics, but much more difficult to manufacture in the real world! If you wish, you can imagine it as an air hockey table, or a particularly smooth and slippery sheet of ice or teflon.) A hockey puck is sliding along the plane, where it hits a big clock hand, and *sticks* to the end of the clock hand. The other end of the clock hand is nailed into the ground, so that it's not going anywhere. After the hockey puck hits the clock hand, the clock hand starts spinning around.



On first glance, you might think, wait! Momentum isn't conserved here! The clock hand may be spinning around, but it's no longer moving off in one direction, whereas before there was clearly momentum in the x-direction! However, remember that the clock hand is nailed into the ground. That means when the puck collides with the clock hand, the clock hand will push on that nail, which pushes on the ground, and effectively the *whole earth* is pushed off (very, very slowly!) to the right. Momentum is conserved, but you have to consider *everything* that's interacting to keep track of all of it.

So, let's do another experiment. Let's collide the puck with a bar— and still have it stick— but not nail that bar to the ground. What happens now is that after the collision, the bar *does* move off to the right only not as fast. It's still rotating, though.

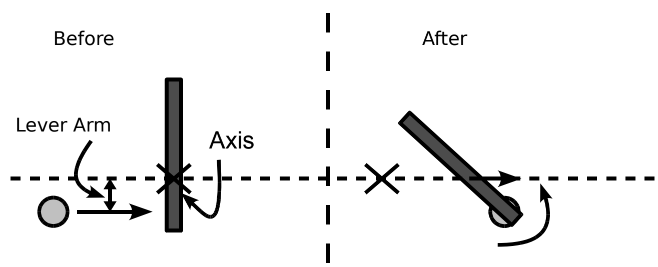


Here, momentum is conserved. The combined system moves off to the right at a lower speed than the puck came in because it's a more massive system. However, there's also the rotation. There's clearly some kinetic energy associated with that, as bits of the rod have motion about the center of the rod in addition to the bulk motion of the rod as a whole. It turns out that there is yet another quantity, called *angular momentum*, that is conserved in interactions. Be careful about the name! Angular momentum is not a "special kind" of momentum. It is a wholly different quantity, with different units, that is conserved separately. It bears some similarities to momentum, and thus the name is similar, but it is a different thing. You cannot mix momentum and angular momentum; again, remember that they are *two different things*.

Back to our example here: the question is, if there's angular momentum afterwards in the rotation of the rod, what angular momentum is there before the collision? It must be there, if angular momentum is a conserved quantity!

In order to measure angular momentum, you must choose an axis to measure it about. How can angular momentum be a conserved quantity if you can choose any arbitrary axis you may ask? The answer is that angular momentum is conserved about *any* axis, as long as you stick with the same axis all the way through the problem.

If a particle is moving in a straight line directly towards or directly away from your chosen axis, then it has *no* angular momentum. However, if it's motion is offset from the axis, even if it's moving in a straight line, it still has angular momentum. To figure out the angular momentum, you multiply the *lever arm* by the momentum of the moving particle. The lever arm is the *perpendicular* distance from the axis to the line of motion of the particle. For example:



I've cleverly chosen my axis to be on line with the motion of the center of mass of the system after the collision. That means that the after the collision, the linear motion of the center of mass of the system makes no contribution to the angular momentum; all of the contribution comes from whatever the rotation is doing. Before, however, the linear momentum of the puck does contribute angular momentum. The lever arm is, as drawn, the perpendicular distance from the axis to the line of motion of the particle. If we call that perpendicular distance d, then the angular momentum (for which we traditionally use the letter l) is:

$$l = dp = dmv$$

where m is the mass of the puck and v is the initial speed of the puck.[1]

How do you figure out the angular momentum of a rotating object? The hard way to do it is to consider the object as a collection of a lot of little pieces of object. For each small piece of that object, you multiply the small mass of that piece by the speed of that piece resulting from the rotation by the lever arm from the axis to that piece. Add up what you get, and you have the object's angular momentum. In practice, for most objects we're able to define a single number that we call the *moment of inertia*, which takes care a of a lot of that for you. This is a quantity that adds up all of the bits of mass and the distances of those bits of mass from a specified axis of rotation for an object. It takes into account the mass part of momentum and the lever arms for all those little bits of the object. The angular momentum of an object rotating about a given axis is then:

$$l = I\,\omega$$

where I is the moment of inertia of that object about the axis and ω is the *angular speed* of the rotation. To figure out angular speed, first figure how how long it takes for the object to make one complete rotation; call that the period T. The angular speed is then:

$$\omega = \frac{2\pi}{T}$$

ω then has dimensionality of one over time; the SI unit for ω is s^{-1}.

3.3.1 The Units of Angular Momentum

If you look at the equation

$$l = d\,p$$

where l is angular momentum, d is the lever arm to a moving particle, and p is the magnitude of the momentum of that particle, you can figure out the units of angular momentum. d has dimensionality of length, of course, and as we worked out in Section 3.2.1, the dimensionality of momentum is mass times length divided by time. Thus, angular momentum has dimensionality mass times length squared divided by time. The SI unit for angular momentum doesn't have a special name; it's just $\mathrm{kg\,m^2\,s^{-1}}$.

3.3.2 The Direction of Angular Momentum

Just like momentum (sometimes called "linear momentum" when you want to be clear that you're not talking about angular momentum), angular momentum is a 3-

[1] If you are familiar with vectors, in fact the real definition of angular momentum is $\vec{l} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the displacement from the axis to the position of the moving particle and $\vec{p}$ is the particle's momentum.

vector. With regular momentum, it's pretty easy to figure out what the direction of the 3-vector is: it's the direction that the object is moving. What, however, is the direction of angular momentum? If an object is spinning, it assuredly has angular momentum. However, the bits of the object are all moving in different directions (the bits on one side of a rotating disk are moving in the opposite direction from the bits on the far side), and what's more, later any given bit of the object will be moving in a different direction.

It turns out there *is* a unique direction for rotation: the axis about which an object is rotating. As such, we can define the direction of the angular momentum 3-vector to be pointing along the axis of rotation. If a Frisbee is flying through the air, rotating, and is parallel to the ground, you would say that its angular momentum 3-vector points either up or down.

How do you figure out up or down? This is just a matter of convention. The convention we use is called the *right-hand rule*. What you do is curl the fingers of your right hand so that they point around in the direction of the rotation. Stick your thumb straight out, and it points along the direction of the angular momentum 3-vector. For example, if you're looking down on a Frisbee, and the Frisbee is rotating counter-clockwise, you would say that it's angular momentum 3-vector is pointing straight up. (Try using your right hand to see why that would be the case.)