Chapter 4

The Spin-$\frac{1}{2}$ particle

Moving electric charges, or currents, interact with magnetic fields; they both respond to them, and create them. You know from Section 3.3 that a spinning ball has angular momentum. If that spinning ball is also charged, that means that, effectively, there are currents associated with the ball. Suppose that the charge is spread uniformly throughout the ball. The charges right along the axis aren't moving, and so wouldn't respond to or create magnetic fields. However, all of the bits of ball that aren't right along the axis are making a circle around the axis. As such, they are moving charges, and they will respond to a magnetic field.

This may seem like a completely unfounded leap, or it may seem like an obvious leap, but from this observation, we can say that a particle that has both charge and angular momentum will respond to magnetic fields.

4.1 Particles in Quantum Mechanics

When we talk about a "particle" in quantum mechanics, we mean something that behaves as if it were just a single body. However, we are often also talking about a particle as it is understood in the Standard Model of Particle Physics. In the Standard Model, a fundamental particle is something that is effectively a mathematical point. As far as we can tell, the fundamental particles have *no* spatial extent. The most common everyday example of a particle from the Standard Model is the electron. You may be familiar with electrons if you have taken any chemistry classes in the past. Atoms are made of of electrons orbiting nuclei. Nuclei themselves are made up of protons and neutrons. Protons and Neutrons may be treated as particles in quantum mechanics, but in fact they are not fundamental particles. Rather, they are themselves made up of *quarks*, which are (at least as far as we understand) fundamental particles.

If a fundamental particle doesn't have a size, what can it have? Well, it can have a position, and it can have a momentum. Later, we will find out that there must be some uncertainty associated with one or both of these quantities for any given particle, but these are quantities that you can figure out for the particle. However, they aren't really fundamental to the particle; they just say where the particle is, or, effectively, how fast it's moving relative to something you've chosen to measure speeds relative to. Similarly, if the particle is an electron in an orbital in an atom, it can have angular momentum as a result of that orbit. Again, this isn't a fundamental property of the particle, but there result of its interaction with the atomic nucleus.

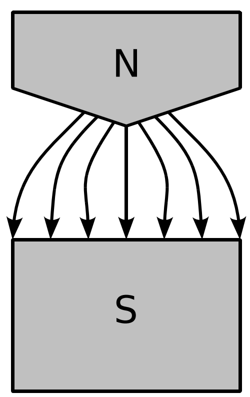
The mass of the particle *is* a fundamental property of the particle. Likewise, the electric charge of the particle. The electric charge on the electron, in SI units, is -1.602×10^{-19} C. In fact, often when we are dealing with atomic and subatomic particles, we'll measure charge in terms of the elementary charge e, which is defined as the absolute value of the charge on the electron: $e = +1.602 \times 10^{-19}$ C. (It is unfortunate that the notation for the elementary charge is the same letter as e, the natural exponential that shows up, for instance, in the mathematical model for radioactive decay. You need to be careful about the context whenever you see an e, so that you can figure out whether we're talking about the natural exponential, the charge on the electron, or something else..)

Another property of fundamental particles is their angular momentum. Because this is fundamental to the particle itself, we refer to it as the *spin* of the particle. As an analogy, consider the Earth orbiting the Sun. The Earth has *orbital* angular momentum as a result of the circle it makes yearly about the Sun. It also has *spin* angular momentum as a result of its daily rotation about its own axis. Where the analogy breaks down, however, is that the Earth is indeed an extended ball; the electron, on the other hand, is a point particle, and has no spatial extent. As such, there really isn't anything spinning around anything else to create this angular momentum. This is conceptually difficult; how, then, can the electron have angular momentum? Alas, the best answer we can give is that it just does. Experiments have shown that indeed electrons behave as if they have angular momentum, and that they can transfer angular momentum to other particles and systems when they interact with them.

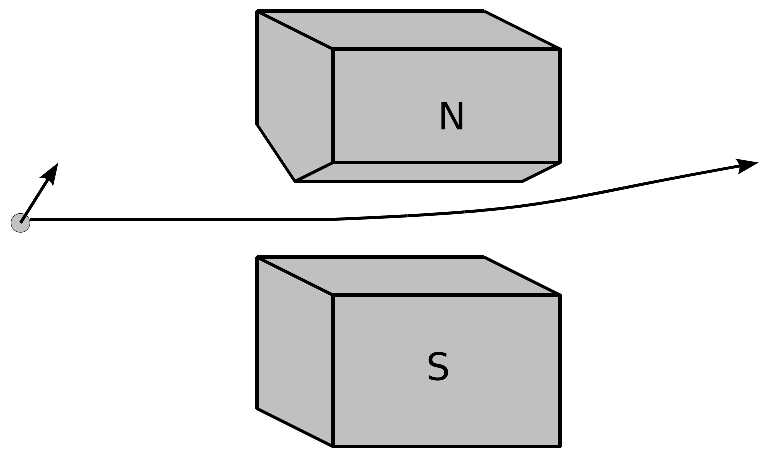
Just as every electron has exactly the same mass and exactly the same electric charge, every electron has exactly the same total angular momentum. (We will see later what the value of that angular momentum is.) You can't cut off a piece of an electron to leave behind a particle that is a part of an electron, with a lower mass and possibly a lower electric charge. Similarly, you can't speed up or slow down the spin of an electron, the way you can get a top spinning faster or slower. All electrons are effectively spinning at the same rate— only, remember, they're not really little balls spinning at all, but rather angular momentum is just one of the properties associated with those quantum particles we call electrons.

4.2 Measuring Electron Spin: the Stern-Gerlach Experiment

If a particle that has both charge and angular momentum interacts with magnetic fields, and if we know what that charge is through other experiments, then we ought to be able to figure out the angular momentum of that particle by some sort of experiment involving magnetic fields. If a particle with charge and angular momentum moves through a *nonuniform* magnetic field, it will be pulled along the direction of the nonuniformity based on the projection or component of its angular momentum along the direction of the magnetic field nonuniformity.



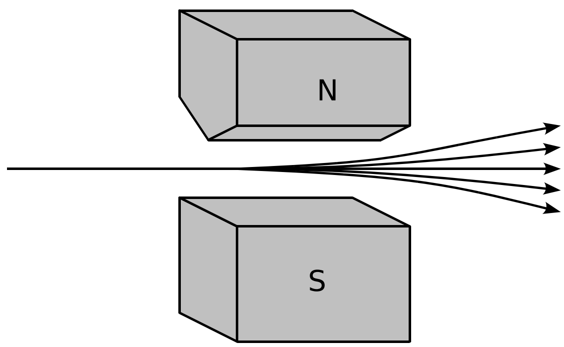
A nonuniform magnetic field, as seen by a particle that will be shot into the page through it.



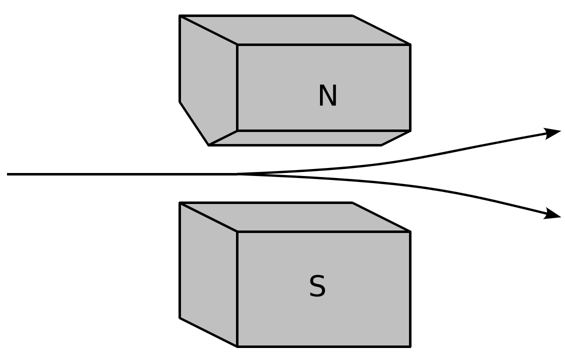
A charged particle with some component of angular momentum along the direction of the nonuniform magnetic field will have its path bent by that field. Whether the path bends up or down depends on the charge of the particle and the direction of the angular momentum.

Remember that angular momentum is a 3-vector. For a spinning object, the angular momentum 3-vector is oriented along the axis about which the object is spinning. To figure out which direction along that axis the angular momentum points, you use the right-hand-rule: orient your *right* hand so that if you curl your fingers, they point along the sense of rotation. Then, your thumb points along the direction of the angular momentum 3-vector. For a classical spinning object like a top or a planet, that angular momentum 3-vector can point in any direction. Indeed, the angular momentum 3-vector of the Earth's rotation is pointed at an angle of 23.5° with respect to the angular momentum 3-vector of the Earth's orbit; they're not perfectly aligned.

Let's imagine what a classical physicist, having accepted (somehow) that all electrons have exactly the same angular momentum, would expect to see if he sent a beam of electrons through a nonuniform magnetic field that bent electrons along the z-axis. If an electron's angular momentum happened to be oriented entirely along the $+z$-axis, its path would be deflected upwards the maximum amount. If its angular momentum happened to be oriented entirely along the $-z$ axis, its path would be deflected downwards the maximum amount. Most of the electrons would have their angular momentum 3-vector randomly oriented somewhere in between, and so the beam should spread out into a vertical smear as it passed through the nonuniform magnetic field.



In the early 1920's, two physicists, Otto Stern and Walther Gerlach, performed this experiment.[1] What they observed was not a continuous smear, but rather that the beam split into two different beams.



Think about what this means. This means that when you take a beam of electrons whose angular momenta are all randomly oriented, if you measure the z component of angular momentum you get one of *only two different values*. The component of spin angular momentum of an electron along the z-axis is either $5.27 \times 10^{-35}\,\mathrm{kg\,m^2\,s^{-1}}$, or $-5.27 \times 10^{-35}\,\mathrm{kg\,m^2\,s^{-1}}$. The z-component of the spin angular momentum of the

[1]Stern and Gerlach *did* measure the spin of the electron, but at the time they *thought* they were measuring quantized orbital angular momentum! For the history of this experiment, see Bernstein (2010).

electron is *quantized*. These values of angular momentum relate Planck's constant $\hbar$ (pronounced "h-bar"), which has the value $\hbar = 1.055 \times 10^{-34}\,\mathrm{kg\,m^2\,s^{-1}}$. When the z-spin of an electron is measured, it comes out to either $+\hbar/2$ or $-\hbar/2$. Indeed, it becomes much more convenient to measure angular momentum in units of $\hbar$ in quantum mechanics, so we refer to the electron as a "spin-$\frac{1}{2}$ particle". Remember, however, that whenever somebody says that an electron is measured to have z spin of 1/2, they really mean that the z component of its angular momentum is $+\hbar/2$.

We define an *observable* as a quantity that we could, at least in principle, measure. The position of a particle is an observable, as is its momentum. The z component of the spin angular momentum of an electron is an observable. One of the primary features of quantum physics is that many observables have the same property that we see for electron spin: when they are in fact observed, they take on one of a finite number of values. They are quantized. It is this property from which quantum mechanics takes its name.

In Quantum Mechanics, many observables are quantized. That is, when measured, they take on one of a finite number of possible values.

It's tempting to think of the electrons whose z spins are $+\hbar/2$ as having their angular momentum oriented entirely along the $+z$-axis, and those whose z spins are $-\hbar/2$ as having their angular momentum oriented entirely along the $-z$-axis. Indeed, physicists will often refer to "spin up" and "spin down" particles. However, the *total* angular momentum of an electron is actually $(\sqrt{3}/2)\hbar$. That means that you *never* observe an electron with its spin oriented entirely along the z axis! There must always be some component of spin oriented along another axis.

4.2.1 The Stern Gerlach Machine

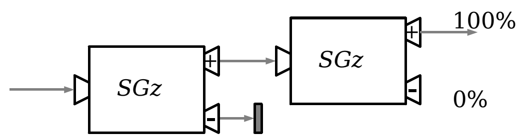
As we continue to explore electron spins in quantum physics, we're going to use a measuring device that repeats the Stern-Gerlach experiment so often that it's worth describing an imaginary "Stern-Gerlach machine". Such a machine has a single input, into which you send a beam of electrons (or even just a single electron). It has two outputs, one for electrons whose angular momentum has been measured as positive along the axis of the machine, the other whose angular momentum has been measured as negative along the axis of the machine. There's no reason why the Stern-Gerlach experiment has to measure the z component of electron spin. By rotating the magnets used in the device, you could measure the x component or y component of the spin. (It's trickier to measure component of spin along the direction of motion of the particle, but that can be done as well.) We will draw an SG machine as follows:

Each time we have an SG machine, it will be named such that the third letter tells you the axis along which its measuring the angular momentum. Thus, an SGz machine measures the z spin of an electron, and an SGx machine measures the x spin of an electron. You could also imagine an SG machine that has its axis oriented at some other angle θ with respect to the z axis. (If that angle is 90°, then it's an SGx machine.) In that case, we will call it an SGθ machine.

4.3 Repeated Measurements of Spin

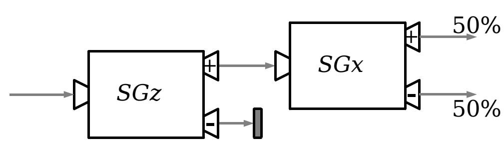
If you have a beam of electrons with randomly oriented spins, when you measure the z spin of the beam you get half of the electrons showing a spin of $\hbar/2$ and half showing a spin of $-\hbar/2$.

Suppose that you block off the beam with negative z spin. Send the beam with positive z spin into a second Stern-Gerlach machine. What do you get?



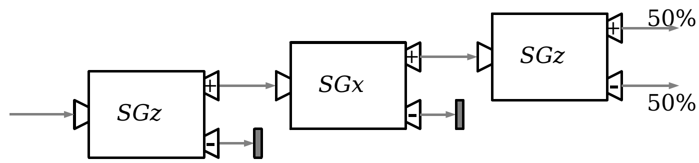
Unsurprisingly, the second SGz machine shows that every electron that comes into it has $+z$ spin. You wouldn't expect anything else. After all, we divided up the electrons that went into the first SGz machine based on their z spin, and threw out the ones that didn't have $+z$ spin.

What if you put the $+1/2$ out put of the SGz machine into an SGx machine?



When the x component of the spin angular momentum of an electron is measured, just as with the z component it only takes on values of $+\hbar/2$ or $-\hbar/2$. Because the x axis is perpendicular to the z axis, you wouldn't expect knowing whether the z spin of the electron was along the $+z$ or $-z$ direction to tell you anything about the whether the x spin was positive or negative. And, indeed, that's what's observed. For each electron with spin $+z$ that goes into an SGx machine, there's a 50% chance it will come out the $+x$ output, and a 50% chance it will come out the $-x$ output.

Things get interesting when you add one more SG machine to the mix. Take the electrons that were first measured to have $+z$ spin, and were then measured to have $+x$ spin. That is, at the first SG machine (an SGz machine), we're throwing out the electrons with $-z$ spin, and at the second SG machine (an SGx machine), we're throwing out the electrons with $-x$ spin. What happens if you send these electrons through another SGz machine? You might expect all of them to come out through the $+z$ output; after all, we already know from a previous measurement that all of these electrons have a positive z component of spin angular momentum. In fact, however, this is *not* what's observed! If you construct this experiment, you find that the final SG machine, an SGz machine, puts out electrons through either output with a 50% chance for each!



The fact that angular momenta were quantized was the first thing about quantum mechanics that was completely at odds with our intuition and our experience with classical physics. This is the second thing. It seems that, somehow, by measuring the x spin of the electrons, we *lost* information about the z spin of the electrons. To explain this and similar experiments, the theory of quantum mechanics includes formalism that shows that it is impossible to know certain pairs of observables at the same time. This is related to the famous Heisenberg Uncertain Principle, about which we will say more in a later chapter. If you know the z spin of an electron, you know nothing about its x spin; were you to measure the x spin, you have a 50% chance of measuring either $+1/2$ or $-1/2$. Likewise, if you know the x spin, you know nothing about its z spin.

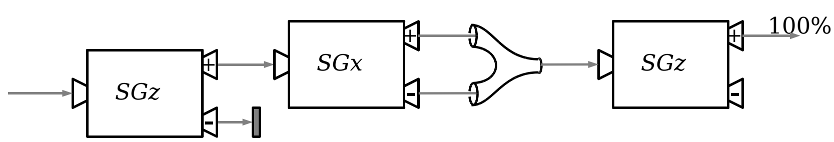
The same result is observed if, instead of the $+x$ output, we take the $-x$ output of the second machine. We have a beam of electrons who all were first measured to have positive z spin, and were then measured to have *negative* x spin. As before, if we measure the z spin again, we find that we have a 50% chance of measuring $+z$, and a 50% chance of measuring $-z$.

The quantum weirdness goes deeper than that. It turns out that it's not just that you *don't know*. The particles themselves do not have a definite state! If you've measured the z spin of an electron, the electron *does not have a definite x spin*! The jargon we use to describe this is to say that the electron is in an "indefinite state", or that it is in a "mixture of states". In this case, the x spin state of the electron is a mixture of the $+1/2$ and $-1/2$ states.

In Quantum Mechanics, certain observables may not be known— do not even take on definite values— at the same time as certain other observables.

At this point, you might object, reasonably so, that we could have neglected an effect of our measuring devices. Charged particles with angular momenta interact with magnetic fields. Could it not be that our devices aren't only deflecting the electrons' paths, but also rotating those electrons? That is, after the first SGz machine, the electrons coming out of the $+z$ output do have z spin of $+1/2$. But when they go through the SGx machine, perhaps it's not just *measuring* the x spin, but also rotating the electrons so that their angular momenta no longer as up along the z axis as they were before. Indeed, it's clear that the state of the system is changed when the x angular momentum is measured. Must it really be something particular to quantum mechanics?

To answer that question, suppose that after we've sent the beam through the SGx machine, dividing it into a beam of electrons with positive x spin and a second beam with negative x spin, we recombine those two beams. Take the recombined beam and put that into the third SGz machine. What do we observe?



The beam *did go* through the SGx machine, so any effect it has on the beam has happened. Remember that the beam coming out of the $+1/2$ output of the SGx machine had an indeterminate z spin; likewise for the beam coming out of the $-1/2$ output of the SGx machine. Yet, somehow, by recombining the beams, we are able to restore the information about the z spin of the electrons! Again, if we make it so that the beam has a very low intensity, and only one electron is going through the apparatus at a time, exactly the same result is observed. In a sense, by recombining the beams, we never really did measure the x spin of the electron. Sure, the SGx machine measured it. . . but we never let the measurement go beyond that, we never let it go into any other experimental apparatus, we didn't let any physicists know about it, we didn't record the spin of any given electron.

There is something peculiar about *measurement* that changes the state of a system. Yet, exactly what is a measurement is not entirely clear. Indeed, the "measurement problem" in quantum mechanics has troubled physicists for nearly a century, and remains a point of active debate today. We will discuss this in greater detail in a later chapter.

For the time being, however, review the results of the various experiments combining SG machines together. The set of observations that we see can not be explained by pure classical physics. The fact that particles have quantized values is already unfamiliar enough. Add to that the fact that for some pairs of observables, such as the z and x components of angular momentum, you can't know both observables at the same time. Finally, on top of all of that, you can destroy, but then somehow reconstruct, information about the state of a given observable based on whether there are multiple paths a particle could have followed, and how those paths are put together.

In future chapters, we will explore the mathematical formalism that physicists have developed to model this behavior.