# Chapter 5

# Dirac Notation

## 5.1 The State of a System

The *state* of a system means the collection of all properties that that system may have. For example, consider an electron. If you wanted to specify the state of that electron as completely as possible, you'd have to specify where it is and its momentum, as well as how well determined its momentum is. You would also have to specify the state of its angular momentum. Does its $z$ spin have a definite or an indeterminate value? If it has a definite value, what is it? If it has an indeterminate value, is it a half and half chance that, if measured, you'll get +1/2 or -1/2, or is it more likely to be one or the other? If your system includes more than just one particle, you have to include all the information about other particles, as well as any information that arises as a result of the interaction between the particles. For instance, if this electron is moving in some potential, for instance because it's part of an atom, what (effectively) is the electron's potential energy?

We are going to introduce an abstract mathematical notation that will indicate "the state of a particle". The notation itself won't necessarily have all of the information above. However, what it will give us is a way to talk about the state of a particle. Because the state of a particle potentially includes a lot of information, it will be necessary to use a more abstract notation than you're used to for mathematical objects. However, remember that even the seemingly-concrete math that you're comfortable with itself is just constructed from abstract mathematical representations of reality.

Consider algebra. Suppose you have a variable, that may or may not be known. You use the name $x$ to represent the state of that variable. Now, if we're dealing with algebra, and we're dealing with only real numbers, then it's possible to represent the full information about the state of this variable with just a single number. For instance, suppose you are given the following algebraic equation:

$$2x + 5 = 9$$

You could use the rules of manipulating algebraic equations to determine that $x = 2$. At that point, you know everything there is to know about the state of this variable. However, you could still represent it with the letter $x$ if you wished to. Even if you don't, however, and if this equation is supposed to represent something from the real world (say, ages of children in a word problem), even the 2 is a mathematical representation of something in the real world.

Let's make it more abstract. Suppose I tell you that $2x + y = b$, and that $y$ and $b$ are known. However, I haven't given you a number to fully specify $y$, nor have I given you one for $b$. I then ask you what x is. You could solve this and tell me that

$$x = \frac{b - y}{2}$$

Now you would say that $x$ is "known", even though you can't reduce it to the concrete representation of real numbers. However, you have given me a representation of $x$ in terms of other things, including this letter $y$ and this other letter $b$. Those two are stand-ins, abstract mathematical representations of "some number that we've decided to call $y$" and "some number that we've decided to call $b$".

We will use a similar abstract notation to represent "the state of a quantum particle", or, perhaps, just "the angular momentum state of a quantum particle" (if we don't care about things like position and momentum). The rules of quantum mechanics will give us mathematical operations we can perform on this representation, and then other things we can do to extract useful information out if it (such as the energy of a particle, or the probability that its $z$ spin will be positive if measured).

## 5.2 The Ket Vector

To represent the state of a quantum particle, or a quantum system, we introduce the "ket vector"

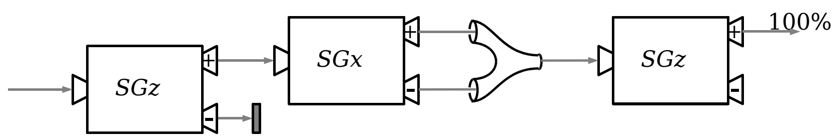
$$|\psi\rangle$$

This is just an abstract mathematical notation, a compact way of saying "the state of this particle". The name "ket" is the latter half of the word "braket", a misspelling of bracket. We will later learn about "bra" vectors, which are written as $\langle\psi|$. However, don't worry about those for now; let's focus on Schrödinger's kets.

Why the $\psi$ inside the ket vector? It's traditional to use the Greek letter "psi" for the state of a system. However, you could put anything you want inside the vertical bar and right angle bracket. It's similar to the convention of using $x$ as the variable for the horizontal axis of a graph in algebra or geometry; people do it a lot, but you can use any letter you want. Sometimes, we will use other Greek letters. Sometimes, we will use something that gives useful information about just what this state is. However, even if we just use something that doesn't tell you anything, like $\psi$, remember that this ket vector is just a way of representing the state so that we can talk about it, so that we can get a handle on it, and so that we can perform mathematical operations with it.

Why do we call it a vector? This is potentially a source of confusion. This is *not* a vector in there-dimensional space, the way angular momentum, velocity, momentum, or displacement are. In fact, it's a vector in an abstract mathematical space called a "Hilbert space". However, for now, don't worry about that. We will see later the ways in which the state vector behave sort of like three-dimensional vectors like velocity. For now, take it as an idiom that when we talk about the "state vector", we're just talking about a mathematical representation of the state of a quantum particle or a quantum system.

Consider, for example, the sequence of Stern-Gerlach machines shown below:




Consider an electron going into the first SG$z$ machine. If its angular momentum can be oriented in any direction, we would say that the electron is "unpolarized". We don't know the state of the electron, so we will just pick a name for it, and call it $|\psi\rangle$.

Now consider an electron coming out of the first SG$z$ machine. If it is measured to have a $z$ spin of $+\hbar/2$, then it will come out of the + output. At that point, we know the $z$ angular momentum of the electron, so let's wisely choose a representation for the state that will make it easy for us to remember: $|+z\rangle$. Note that the "$+z$" inside the notation doesn't mean anything about adding any variable named $z$, nor have we defined a variable $z$. It's *just a name*. I could just as easily have named the state vector $|\text{Fred}\rangle$. That would have allowed us to carry it around in equations, talk about it, and perform mathematical operations with it. However, for us humans reading the equations, it's convenient of the name is something that reminds us what we know about the state. So, we'll choose $|+z\rangle$ as our name so that we remember that, aha, this is an electron whose $z$ spin is known to be positive.

Similarly, an electron coming out of the $-$ output from the SG$z$ machine will be in the state $|-z\rangle$. Again, there's no subtraction, or any multiplying by negative one going on here. It's just a name, the same way $x$ is just a name for a (possibly unknown) variable in algebra.

Now move on to the second SG machine, the SG$x$ machine. An electron going into this machine is in state $|+z\rangle$; we know that, because all of these electrons are coming out of the + output of an SG$z$ machine. However, as this electron goes through the SG$x$ machine, its state changes. If it comes out the + output of the SG$x$ machine, it's in a state we shall choose to call $|+x\rangle$. If it comes out of the − output of the SG$x$ machine, it's in a state we shall choose to call $|-x\rangle$.

Remember what happened in the previous chapter when we then took an electron in state $|+x\rangle$— that is, an electron whose $x$ spin was measured to be positive— and put it back into a second SG$z$ machine. It had a 50% chance of being measured with positive $z$ spin, and a 50% chance of being measured with a negative $z$ spin. In other words, the electron is no longer in state $|+z\rangle$, nor is it in state $|-z\rangle$; the state $|+x\rangle$ is different from both of those $z$ states.

It turns out, however, that you *can* describe the state $|+x\rangle$ in terms of the states $|+z\rangle$ and $|-z\rangle$. Remember the algebraic equation $2x + y = b$, which allowed us to figure out that $x = (b - y)/2$. The variables $b$ and $y$ are abstract representations of numbers, and $x$ is an abstract representation of another number. The equation $x = (b - y)/2$ tells us that the thing that $x$ represents is not independent from $b$ and $y$; were $b$ or $y$ to change, $x$ would have to change along with it. It also tells us how to figure out $x$ in terms of $b$ and $y$. If we have rules for doing things to $b$ and $y$, we can then apply those rules to the right side of the equation to figure out how $x$ changes.

Bearing that in mind, it turns out that you can represent the $x$ spin states of an electron in terms of the $z$ spin states as follows:

$$\begin{aligned} |+x\rangle &= \tfrac{1}{\sqrt{2}}|+z\rangle + \tfrac{1}{\sqrt{2}}|-z\rangle \\ |-x\rangle &= \tfrac{1}{\sqrt{2}}|+z\rangle - \tfrac{1}{\sqrt{2}}|-z\rangle \end{aligned}$$

In a future chapter, we will see why this seemingly odd combination, with all of its square roots of two, makes sense. We will also see why being able to represent the $x$ states in terms of the $z$ states is useful. For now, however, recognize this as the first rule about performing mathematical operations on these state vectors: **you can multiply a state vector by a number**. Don't worry about how you would actually calculate something from that. In algebra, you can write down $x/2$, it has meaning even if you don't know $x$ and can't calculate a number for $x/2$. Or, if you're doing algebra with 3-vectors in space, you could write down $\vec{v}_1 = \vec{v}_2/2$. Even if you don't have numbers for all three components of $\vec{v}_2$, and thus can't calculate numbers for all three components of $\vec{v}_1$, this equation is still meaningful. Similarly, it's a valid mathematical operation to multiply a ket vector by a number. For now, we'll just leave it written out as that number multiplied by the ket vector, and bear in mind that when you multiply a number (a *complex* number— it doesn't have to be real!) by a ket vector, you get another ket vector as a result.

You can also add two ket vectors together, and the result of that addition is yet a third ket vector. Here, we see that particular combinations of constants times $|{+z}\rangle$ and $|{-z}\rangle$ turn out to be equal to $|{+x}\rangle$. This is a mathematical expression, within the theory of quantum mechanics, that represents a truth about reality. It's very similar to the Pythagorean Theorem, $a^2 + b^2 = c^2$, which expresses a truth about triangles (if $c$ is a representation of the length of the hypotenuse of the triangle, and $a$ and $b$ are representations of the lengths of the legs of the triangle).

Let's go through the next step of our sequence of devices above. After the SG$x$ machine, the two beams (one in state $|{+x}\rangle$ and one in state $|{-x}\rangle$) are recombined together into a single beam. Let us pretend that we don't know what the state of the electron coming out of the recombining apparatus is. Just as in algebra, when we have a quantity we don't know, we'll give this state a name; let's call it $|\psi_{\mathrm{RC}}\rangle$, with "RC" standing for "recombined". If we knew the all of the rules for figuring out how quantum states evolve as they pass through these SG machines, we could figure out what this state is by performing calculations on the previous states, based on where the beams have been. However, we don't yet know these rules. Instead, what we do is perform one more thought experiment: we send this electron, in state $|\psi_{\mathrm{RC}}\rangle$, through an SG$z$ machine. We discover that 100% of the time, the electron comes out of the + output of the SG$z$ machine. From this experiment, we've figured out that

$$|\psi_{\mathrm{RC}}\rangle \;=\; |{+z}\rangle$$

In upcoming chapters, we'll learn how to figure out theoretically that this is the state of that electron.