Chapter 6

Amplitudes and Probabilities

The classical physics represented by Newton's Laws is deterministic. The equations tell you that if a particle is *here* and its speed is exactly just *this much*, then it *will* be over here and moving this fast later. It gives us a picture of a clockwork universe, where everything future possible measurement is completely determined by the current state of the system.[1]

In quantum physics, as we have seen, this is not the case. If you have an electron whose spin has been measured to be pointing along the $+z$ axis, then the best statement you can make about the x projection of the electron's spin angular momentum is a probabilistic one: there is a 50% chance you'll measure x spin along the $+x$ direction, and a 50% chance you'll measure x spin along the $-x$ direction. What's more, this probabilistic nature is not simply due to our lack of knowledge. Statistics is an entire branch of mathematics used to estimate what we know and determine our confidence in what we know when we have imperfect information. While statistics does apply to quantum mechanics, most of the time statistics is employed in practice the probabilities come not from a fundamental probability, but from lack of perfect knowledge about the state of the system, or because the system itself contains individuals who vary. In quantum mechanics, this probabilistic nature runs more deeply, even though each and every electron is identical. Whereas in classical physics, we may never be able to make perfect measurements, but the theory underneath them is able to presume perfectly determined quantities. In quantum mechanics, the theory needs to be able to handle the calculation and propagation of these probabilities.

[1] In fact, chaos theory has shown us that nonlinearities even in classical physics place a limit on the predictability of those systems. However, the laws themselves *are* deterministic.

6.1 Complex Numbers

Before we begin, however, we need briefly to review complex numbers. Complex numbers are intrinsic to quantum mechanics, and indeed the entire theory wouldn't work if we didn't use complex numbers as part of it.

A complex number is a number that has both a real and an "imaginary" part. The name "imaginary" is perhaps unfortunate, because it suggests there's something less tangible about imaginary numbers than there is about real numbers. Remember, however, that even real numbers, when used in science, are abstract mathematical representations of the systems that they are standing in for. Even real numbers are imaginary, in that sense of the word.

All imaginary numbers can be constructed from i, sometimes called "the" imaginary number, which is defined as:

$$i = \sqrt{-1}$$

you may remember from math that you can't take a square root of a negative number. In fact, you can, but you don't get a real number as a result; you get an imaginary number. By the same token, you may remember that the square of any number is positive. That only applies to real numbers; the square of any real number is positive. However, square both sides of the equation above and you can see that:

$$i^2 = -1$$

You can construct any other imaginary number by just multiplying i by a real number. So, $3i$, πi, and $-2.9 \times 10^{21} i$ are all imaginary numbers.

You can then write any complex number as the real part plus the imaginary part. So, $2+3i$ is a complex number. You can't simplify it any further than that. Remember that i is not a variable here, but a number, just as concrete as any other number. It's not a number that you could place on a numberline, because a numberline only has the real numbers on it. But it's just as. . . well, just as real as a real number. The expression $2 + 3i$ is fundamentally different from the expression $2 + 3\pi$. You can view $2 + 3\pi$ as being completely reduced, as there's no need to reduce it further (as there would be with the expression $2 + (3)(4)$, which can be reduced to 14). However, you could, if you wished, reduce $2 + 3\pi$ with your calculator, and write down an imperfect single-valued representation of it: 11.424778. No such further reduction may be done with the number $2 + 3i$. The two parts of this number, 2 and $3i$, are like two components of a vector; they both have an independent identity. However, when we get to using vectors to represent spin states of particles don't confuse the real and imaginary parts of a complex number with components of those vectors. The value $2 + 3i$ represents a *single* complex number. You can reduce the real part and the imaginary part of a complex number down so that the first is represented by a single real number, and the second is represented by a second real number multiplied by i.

For every complex number, there is a partner number called the *complex conjugate*. Along with the noun complex conjugate there is a verb complex conjugate. In order to complex conjugate a number, you replace every instance of i with $-i$. So, the complex conjugate of $2 + 3i$ is $2 - 3i$. In algebra, we use the symbol * to indicate the complex conjugate of a quantity. If you have a complex number a (i.e. a variable in algebra that may not just have a real value, but which may have a fully complex value), the complex conjugate of a is represented as a^*. Thus, if $a = 2 + 3i$, then $a^* = 2 - 3i$.

6.2 Amplitudes

What makes quantum mechanics so different from the propagation of uncertainty in classical physics is that it's not directly the probabilities that propagate, but rather these things called *amplitudes*. Suppose you constructed something like a Stern-Gerlach machine, and propagated the system through it using the rules of classical physics. Suppose the path of the particle has two places where there are two possibilities. Suppose that at each of these branches, the probability of each branch is 1/2. That would leave you with four possibilities in the end. The rules of probability tell you that the chance that a particle will take a certain branch at the first choice, *and* a certain branch at the second choice, means that you have to multiply the probability of each branch at each choice. In this example, that would leave you with an overall 1/4 probability of the particle having gone through a given path.

In quantum mechanics, however, the situation may be entirely different. The probabilities you get at the end cannot be simply calculated from the probabilities you would get if you evaluated each choice in isolation.

There is another realm of mechanics where paths taken by the system depend more directly on amplitudes than on probabilities, and that realm is wave mechanics. If two waves pass each other, it's possible to get destructive or constructive interference, possibly giving you wave intensities that are the sum of the two individual intensities, but also possibly giving you wave intensities of zero. Indeed, it is from the amplitudes of waves that quantum mechanics gets the term amplitudes for the thing that it propagates in order ultimately to calculate probabilities. Quantum mechanics bears a lot of similarities to more general wave mechanics, and indeed we often refer to the state of the system $|\psi\rangle$ as the "wave vector" or the "wave function." Although we will not explore this statement in great detail in this course, it is correct to say that in quantum mechanics, particles often (but not always) behave more like waves than like particles. Different quantum states may *interfere* with each other in the same way that waves can interfere with each other. From this interference arises much of the non-intuitive nature of quantum mechanics.

6.2.1 Calculating Probabilities from Amplitudes

Suppose that you have the amplitude A for a particle to be in a given state. Sometimes, this is all you want. You may need to use it to calculate the interference of this state with another state. However, often, what you really want is the probability P for that particle to be found in that given state. You can calculate P by A by taking the *absolute square* of A, written as $|A|^2$. This is different from squaring A, in that you don't multiply the number A by itself, but rather you multiply A by its complex conjugate. So, if A is the amplitude for a particle to be in a given state, then the probability P for that particle to be in that state is:

$$P = |A|^2 = A^*A$$

As an example, suppose that you've calculated that the amplitude for a particle in state $|\psi\rangle$ to be subsequently measured to have $+z$ spin (and thus go into the state $|+z\rangle$) is $(2+i)/3$. If we wanted to calculate the probability, we'd need to multiply this by its complex conjugate:

$$\begin{aligned} P &= \left(\tfrac{2+i}{3}\right)^* \left(\tfrac{2+i}{3}\right) \\ &= \left(\tfrac{2-i}{3}\right) \left(\tfrac{2+i}{3}\right) \\ &= \tfrac{(2-i)(2+i)}{9} \\ &= \tfrac{4+2i-2i-i^2}{9} \\ &= \tfrac{4+1}{9} \\ &= \tfrac{5}{9} = 0.55555\ldots \end{aligned}$$

6.3 Bra Vectors and the Inner Product

For each ket vector $|\psi\rangle$, there is a corresponding bra vector $\langle\psi|$. We haven't yet looked into any specific representations of ket vectors beyond just the ket vector itself, so at the moment that's all you need to know. However, when we do get into specific representations, the rules for converting ket vectors to bra vectors are generally very easy. You always take the complex conjugate of any numbers in the representation going from the ket vector to the bra vector. (You may also turn a column vector into a row vector, if you're using column vectors to represent ket vectors; much more about that later.) $\langle\psi|$ is *something like* the complex conjugate of $|\psi\rangle$, although that's not really right. However, just as a number and its complex conjugate are associated with each other, each ket vector $|\psi\rangle$ is uniquely associated with a bra vector $\langle\psi|$.

With the introduction of bra vectors, it becomes possible to define a new operation you can do on these things. You can always stick a bra vector on to a ket vector. The notation is meant to help suggest this; where there is a straight side, you can stick two of them together. The result is called the *inner product*. The specific rules for how you calculate the inner product again depend on the detailed representation of the ket vector, so for now we'll keep them abstract. As an example, suppose you have two different quantum states represented by the ket vector $|\psi\rangle$ and the ket vector $|\phi\rangle$. The bra vector corresponding to the latter is $\langle\phi|$, and the inner product of that bra vector with the ket vector $|\psi\rangle$ is:

$$\langle\phi\,|\,\psi\rangle$$

When you see a bra-ket pair combined like that, the result is a **scalar**! It may well be a complex number, but it is just a number. At that point, you can manipulate it in algebraic equations the way you would manipulate any other complex number.

The inner product of a bra and a ket is the first way we've seen to multiply two of these state vectors together. We've talked about multiplying the state vectors by a scalar, but before we didn't know how to multiply them together. Notice, however, that this is a different sort of multiplication than multiplying two scalars. When you multiply two scalars, you get another scalar out— the same sort of thing as the things you multiplied together. However, when you take the inner product of two state vectors, you get a scalar out, something different from the two things that went into the inner product.

Note that you can only take the inner product between two quantum states if they are the same sort of state. That is, they must be the same kind of state for the same particle or system. For instance, you could take the inner product between two angular momentum states for the same electron, but you couldn't take the inner product between an angular momentum state and a position state.

6.4 Normalization and Orthogonality

Although we aren't yet going to learn rules for doing general inner products between state vectors, there are two cases where the inner product of two state vectors produces a simple answer. The first is not intrinsic to the mathematical representation, but rather something we will insist for state vectors that properly represent real physical states. For a complete state vector $|\psi\rangle$ to be a proper quantum mechanical state, it must satisfy the condition

$$\langle\psi\,|\,\psi\rangle \;=\; 1$$

We say that this means that the state vector is *normalized*. It is possible to have non-normalized state vectors. For instance, in the equation

$$|+x\rangle \;=\; \frac{1}{\sqrt{2}}\,|+z\rangle \;+\; \frac{1}{\sqrt{2}}\,|-z\rangle$$

the two parts of the sum on the right side are themselves ket vectors. However, because they are valid state vectors multiplied by a constant, they are not normalized themselves. We will show later that this definition of $|+x\rangle$ is, however, normalized.

The second rule is that state vectors that represent different possible states corresponding to different possible measurements of a given observable must be *orthogonal*. Mathematically, this is expressed as:

$$\langle \phi_1 \,|\, \phi_2 \rangle \;=\; 0$$

if $|\phi_1\rangle$ and $|\phi_2\rangle$ are two different states corresponding to definite states for a given observable. For example, the states $|+z\rangle$ and $|-z\rangle$ correspond to two states of the same observable, specifically, the z component of angular momentum. The first corresponds to that component being measured along $+z$, the second to it being measured along $-z$. The orthogonality condition is then:

$$\langle +z \,|\, -z \rangle \;=\; 0$$

As an example of doing these calculations with a more complicated state, consider the state $|+x\rangle$. If this state is properly normalized, then we should have $\langle +x \,|\, +x\rangle = 1$. Do we? Well, first, we have to construct the bra vector that goes along with the ket vector:

$$\begin{aligned} |+x\rangle &= \tfrac{1}{\sqrt{2}}\,|+z\rangle \;+\; \tfrac{1}{\sqrt{2}}\,|-z\rangle \\ \langle +x| &= \tfrac{1}{\sqrt{2}}\,|+z\rangle \;+\; \tfrac{1}{\sqrt{2}}\,|-z\rangle \end{aligned}$$

Notice that in the case of a compound ket vector, to get the bra vector we just turn all ket vectors on the right side into bra vectors, and replace all the numbers with their complex conjugates (which is trivial here, since all the numbers are real). Now we have what we need to figure out the inner product. Just substitute in our expressions for $|+x\rangle$ and $\langle +x|$:

$$\begin{aligned} \langle +x \,|\, +x\rangle &= \left(\tfrac{1}{\sqrt{2}}\,\langle +z| \;+\; \tfrac{1}{\sqrt{2}}\,\langle -z|\right)\left(\tfrac{1}{\sqrt{2}}\,|+z\rangle \;+\; \tfrac{1}{\sqrt{2}}\,|-z\rangle\right) \\ &= \tfrac{1}{2}\,\langle +z \,|\, +z\rangle \;+\; \tfrac{1}{2}\,\langle +z \,|\, -z\rangle \;+\; \tfrac{1}{2}\,\langle -z \,|\, +z\rangle \;+\; \tfrac{1}{2}\,\langle -z \,|\, -z\rangle \end{aligned}$$

That looks very complicated, but now we can use the orthogonality condition we know is true for the z states, as we've defined them as good states corresponding to the z component of z angular momentum. We know that $\langle +z \,|\, +z\rangle = 1$, $\langle -z \,|\, -z\rangle = 1$,

$\langle -z \,|\, +z\rangle = 0$, and $\langle +z \,|\, -z\rangle = 0$ from normalization and orthogonality. Substitute these in:

$$\begin{aligned} \langle +x \,|\, +x\rangle &= \tfrac{1}{2}(1) \;+\; \tfrac{1}{2}(0) \;+\; \tfrac{1}{2}(0) \;+\; \tfrac{1}{2}(1) \\ \langle +x \,|\, +x\rangle &= 1 \end{aligned}$$

So the state is properly normalized! I leave it as an exercise for the alert reader to show that $|+x\rangle$ and $|-x\rangle$ are orthogonal.

6.5 Interpreting the Inner Product

So far, all we know about the inner product is that for a properly normalized quantum state, the inner product of that state with itself is 1, and that the inner product between two different states corresponding to definite states of the same observable must be zero. But what about the inner product between two arbitrary states? Consider:

$$\langle \phi \,|\, \psi\rangle$$

The interpretation of this is that it is the **amplitude for a particle in state $|\psi\rangle$ to subsequently be measured in state $|\phi\rangle$**. As an example, suppose that we have an electron in the following state:

$$|\psi\rangle \;=\; \frac{3}{5}\,|+z\rangle \;+\; \frac{4i}{5}\,|-z\rangle$$

Suppose we send this electron through an SGz machine. If this state is properly normalized (is it?), then we could work out the amplitude for it to be measured in the $|-z\rangle$ state (i.e. the amplitude for measuring its z-spin to be $-\hbar/2$) as follows:

$$\begin{aligned} \langle -z \,|\, \psi\rangle &= \langle -z|\, \left(\tfrac{3}{5}\,|+z\rangle \;+\; \tfrac{4i}{5}\,|-z\rangle\right) \\ &= \tfrac{3}{5}\,\langle -z \,|\, +z\rangle \;+\; \tfrac{4i}{5}\,\langle -z \,|\, -z\rangle \\ &= \tfrac{3}{5}(0) \;+\; \tfrac{4i}{5}(1) \\ &= \tfrac{4i}{5} \end{aligned}$$

This tells us the amplitude for the electron to be found in the $|-z\rangle$ state. Remember that the *probability*, what we really care about, is the absolute square of the amplitude. That probability is:

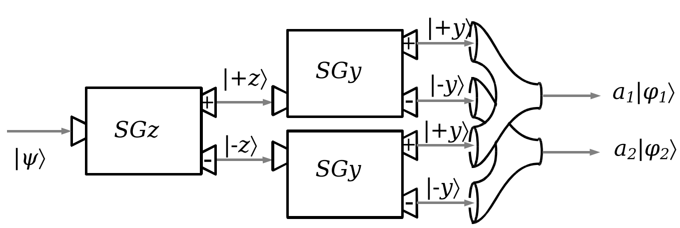
$$\begin{aligned} |\langle -z \,|\, \psi\rangle|^2 &= \langle -z \,|\, \psi\rangle^* \,\langle -z \,|\, \psi\rangle \\ &= \left(\tfrac{-4i}{5}\right)\left(\tfrac{4i}{5}\right) \\ &= \left(\tfrac{-16\,i^2}{25}\right) \\ &= \tfrac{16}{25} \;=\; 0.64 \end{aligned}$$

If the electron was in the state $|\psi\rangle$ defined above upon entering a SGz machine, there's an 64% chance it will come out the $-z$ output of the machine, being measured with a z-spin of $-\hbar/2$.

6.5.1 Propagating Amplitudes

We have seen that the amplitude for a given quantum state $|\psi\rangle$ to later be found in another quantum state $|\phi\rangle$ is $\langle\phi\,|\,\psi\rangle$. Physically, when would you apply this amplitude? You would apply it when the system went through a device that measured whatever quantity $|\phi\rangle$ is associated with. For example, if you have an electron beam in state $|\psi\rangle$ going into an SGz machine, you'd associate the amplitude $\langle +z\,|\,\psi\rangle$ with the state $|+z\rangle$ emerging from the positive output of the machine. What do you do, however, if the electron beam then goes through another machine? How do you deal with amplitudes when there is more than one process that might have a state change associated with it? The answer is that to get the overall amplitude for a starting state to end up in some final state, you multiply the individual amplitudes of each step the system went through.[2]

As an example, consider the following sequence of SG machines:



Yowza.[3] An electron in some state $|\psi\rangle$ goes into the beginning of this system. There are two possible places it may come out. It may emerge from the upper output in state $|\phi_1\rangle$ (which is currently unknown, but we will figure it out); the amplitude for it to emerge here is a_1. It may also emerge from the lower output in state $|\phi_2\rangle$ (which we will also figure out); the amplitude for it to emerge from the lower output is a_2.

Ultimately, what we're interested in is the amplitude for the electron emerging from this whole thing with state $|+y\rangle$, and the amplitude for it emerging with state $|-y\rangle$. To figure those out, we need to trace the electron through all of the possible

[2]This is different from classical physics, where you'd multiply the probabilities. You might wonder what the difference is, since you are going to square the whole thing at the end anyway. The differences comes from the fact that the quantum amplitudes may be *complex*, so the products of the individual amplitudes could end up with terms canceling each other.

[3]The reason why we have this complicated a collection of SG machines and beam combiners is that it's important that we *not be able to figure out* which output from the SGz machine the electron went through, for subtle reasons that will be discussed in the next chapter.

paths. At the input to the first machine, the electron is in the state $|\psi\rangle$. At the positive output of the first machine, the electron is now either in the state $|{+z}\rangle$, with amplitude $\langle {+z} \,|\, \psi\rangle$, or in the state $|{-z}\rangle$, with amplitude $\langle {-z} \,|\, \psi\rangle$.

Let's consider the possible paths for the electron if it comes out of the $+z$ output of the first machine. If the electron goes this way, it will go into the upper SGy machine, with state $|{+z}\rangle$. It will emerge from either the + output, with amplitude $\langle {+y} \,|\, {+z}\rangle$, or from the − output, with amplitude $\langle {-y} \,|\, {+z}\rangle$. The *overall* amplitude for the electron to make it from the very beginning to the + output of the upper second machine is the *product* of the amplitudes for each step: $\langle {+z} \,|\, \psi\rangle \langle {+y} \,|\, {+z}\rangle$. Likewise, the overall amplitude for the electron to make it from the very beginning to the − output for the lower second machine is $\langle {+z} \,|\, \psi\rangle \langle {+y} \,|\, {+z}\rangle$.

Next, consider the possible path of the electron emerging from the − output of the first machine. The amplitude for it to get this far is $\langle {-z} \,|\, \psi\rangle$. The overall amplitude, then, for it to come out of the + output of the lower machine is $\langle {-z} \,|\, \psi\rangle \langle {+y} \,|\, {-z}\rangle$, and the overall amplitude for it to come out of the − output of the lower machine is $\langle {-z} \,|\, \psi\rangle \langle {-y} \,|\, {-z}\rangle$.

What do you do at a beam combiner? There, you just add the two states together, each multiplied by their respective amplitudes. Let's first consider the upper beam combiner. The two states coming into this system, with their respective amplitudes, are:

$$\langle {+z} \,|\, \psi\rangle \langle {+y} \,|\, {+z}\rangle \,|{+y}\rangle$$

and

$$\langle {-z} \,|\, \psi\rangle \langle {+y} \,|\, {-z}\rangle \,|{+y}\rangle \,.$$

Therefore, the final output amplitude and state is:

$$a_1 \,|\phi_1\rangle \;=\; \left[\langle {+z} \,|\, \psi\rangle \langle {+y} \,|\, {+z}\rangle \;+\; \langle {-z} \,|\, \psi\rangle \langle {+y} \,|\, {-z}\rangle\right] \;|{+y}\rangle$$

By looking at this, we can see that the state $|\phi_1\rangle$ is in fact just $|{+y}\rangle$. Hopefully, that does not come as a surprise to you, as the state of the two electron beams going into this beam combiner was just $|{+y}\rangle$. The amplitude a_1 is then just

$$a_1 \;=\; \langle {+z} \,|\, \psi\rangle \langle {+y} \,|\, {+z}\rangle \;+\; \langle {-z} \,|\, \psi\rangle \langle {+y} \,|\, {-z}\rangle$$

The two state going into the lower beam combiner, with their respective amplitudes, are:

$$\langle {+z} \,|\, \psi\rangle \langle {-y} \,|\, {+z}\rangle \,|{-y}\rangle$$

and

$$\langle {-z} \,|\, \psi\rangle \langle {-y} \,|\, {-z}\rangle \,|{-y}\rangle \,.$$

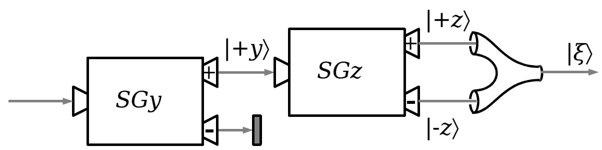
Therefore, the final output amplitude and state for the lower output from this whole system is:

$$a_2 \,|\phi_2\rangle \;=\; \left[\langle {+z} \,|\, \psi\rangle \langle {-y} \,|\, {+z}\rangle \;+\; \langle {-z} \,|\, \psi\rangle \langle {-y} \,|\, {-z}\rangle\right] \;|{-y}\rangle$$

The state $|\phi_2\rangle$ is just $|-y\rangle$, and the amplitude for the lower output is:

$$a_2 \;=\; \langle +z\,|\,\psi\rangle\,\langle -y\,|+z\rangle \;+\; \langle -z\,|\,\psi\rangle\,\langle -y\,|-z\rangle$$

As another example, consider the following collection of SG machines:



We know from the previous chapter that the final state of this system should be $|+y\rangle$. Is that what we get?

In order to analyze this, you're going to need to know how to express the states $|+y\rangle$ and $|-y\rangle$ in terms of $|+z\rangle$ and $|-z\rangle$:

$$\begin{aligned} |+y\rangle &= \tfrac{1}{\sqrt{2}}\,|+z\rangle \;+\; \tfrac{i}{\sqrt{2}}\,|-z\rangle \\ |-y\rangle &= \tfrac{i}{\sqrt{2}}\,|+z\rangle \;+\; \tfrac{1}{\sqrt{2}}\,|-z\rangle \end{aligned}$$

Consider the electron going into the input of the second machine. It is in state $|+y\rangle$. We won't worry about the amplitude for the initial electron to get into this state, because we'll just consider the ones that happen to come out the positive output of the first machine. (The purpose of that first machine is to make sure that we know the electrons are in fact in the $|+y\rangle$ state when they go into the second machine.) The amplitude for an electron to come out of the upper terminal is $\langle +z\,|+y\rangle$, and the amplitude for an electron to come out of the lower terminal is $\langle -z\,|+y\rangle$. Call the final state coming out of the beam combiner $|\xi\rangle$. To figure out what this state is, combine together the two states going into it, each multiplied by their respective amplitudes:

$$\begin{aligned} |\xi\rangle &= \langle +z\,|+y\rangle\,|+z\rangle \;+\; \langle -z\,|+y\rangle\,|-z\rangle \\ &= \langle +z|\,\left(\tfrac{1}{\sqrt{2}}\,|+z\rangle \;+\; \tfrac{i}{\sqrt{2}}\,|-z\rangle\right)|+z\rangle \;+\; \langle -z|\,\left(\tfrac{1}{\sqrt{2}}\,|+z\rangle + \tfrac{i}{\sqrt{2}}\,|-z\rangle\right)|-z\rangle \\ &= \left(\tfrac{1}{\sqrt{2}}\,\langle +z\,|+z\rangle \;+\; \tfrac{i}{\sqrt{2}}\,\langle +z\,|-z\rangle\right)|+z\rangle \;+\; \left(\tfrac{1}{\sqrt{2}}\,\langle -z\,|+z\rangle \;+\; \tfrac{i}{\sqrt{2}}\,\langle -z\,|-z\rangle\right)|-z\rangle \end{aligned}$$

Once again, we just have inner products of z states with themselves. We can use normalization (e.g. $\langle +z\,|+z\rangle = 1$) and orthogonality (e.g. $\langle -z\,|+z\rangle = 0$) to substitute in the numbers from the inner products in the expression above, yielding us:

$$\begin{aligned} |\xi\rangle &= \left(\tfrac{1}{\sqrt{2}}(1) \;+\; \tfrac{i}{\sqrt{2}}(0)\right)|+z\rangle \;+\; \left(\tfrac{1}{\sqrt{2}}(0) \;+\; \tfrac{i}{\sqrt{2}}(1)\right)|-z\rangle \\ &= \tfrac{1}{\sqrt{2}}\,|+z\rangle \;+\; \tfrac{i}{\sqrt{2}}\,|-z\rangle \\ &= |+y\rangle \end{aligned}$$

Sure enough, the mathematical rules for propagating amplitudes has given us the answer that we know is supposed to be right for the final state.